Rule-based Graph Programming

Detlef Plump

University of York, UK
Overview

Part I: Introduction

Part II: GP 2 Foundations
Relabelling
Host graphs
Rule schemata

Part III: Graph Programs
Abstract syntax
Example programs
  Transitive closure
  Graph inverse
  Vertex colouring
  2-colouring
  Shortest distances
  Cyclic graphs
  Series-parallel graphs

Part IV: Operational Semantics
Inference rules
Semantic function
Semantic equivalence

Part V: Verification Case Study
A copying garbage collector in GP 2
Pre-and postcondition
Proof tree
Total correctness

Part VI: Miscellanea
Rooted programs
Other topics

References
Part I

Introduction
Graph Programming Language GP 2

- Experimental domain-specific language for graphs
- Based on graph transformation rules
- Commands to control rule applications
- Non-deterministic
- Simple syntax and semantics to facilitate formal reasoning
Graph Programming Language GP 2

- Experimental domain-specific language for graphs
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- Commands to control rule applications
- Non-deterministic
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Example program: transitive closure

A graph is *transitive* if for every directed path \( v \rightarrow v' \) with \( v \neq v' \), there is an edge \( v \rightarrow v' \).

Program for computing a *transitive closure* of the input graph (smallest transitive extension):

\[
\text{Main} = \text{link}!
\]

\[
\text{link}(a, b, x, y, z: \text{list})
\]

\[
\begin{array}{ccc}
\text{x} & \text{a} & \text{y} \\
1 & 2 & 3
\end{array}
\begin{array}{ccc}
\Rightarrow \\
\text{a} & \text{b} & \text{z}
\end{array}
\]

where not edge(1,3)
Example program: transitive closure (cont’d)
Example program: transitive closure (cont’d)
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Part II

GP 2 Foundations
Graph transformation with relabelling

- How to write a program with double-pushout rules that replaces each node label 1 with 2?
Graph transformation with relabelling

- How to write a program with double-pushout rules that replaces each node label 1 with 2?
- Graph morphisms are *label-preserving*, hence conventional rules cannot relabel nodes:

```
1 ← ? → 2
1 1 1
```
How to write a program with double-pushout rules that replaces each node label \( 1 \) with \( 2 \)?

Graph morphisms are *label-preserving*, hence conventional rules cannot relabel nodes:

\[
\begin{array}{c}
\text{1} \\
\downarrow 1
\end{array}
\quad \leftrightarrow 
\quad \begin{array}{c}
? \\
\downarrow 1
\end{array}
\quad \rightarrow 
\begin{array}{c}
2 \\
\downarrow 1
\end{array}
\]

Rule

\[
\begin{array}{c}
1 \\
\downarrow
\end{array}
\quad \leftarrow 
\quad \emptyset 
\quad \rightarrow 
\begin{array}{c}
2 \\
\downarrow
\end{array}
\]

works only for isolated nodes because of the dangling condition
Graph transformation with relabelling

Solution: *unlabelled* nodes in the interface:

\[
\begin{array}{c}
\{1\} \\
1 \\
\end{array} \quad \quad \quad \quad \quad \quad \\
\begin{array}{c}
\{1\} \\
\end{array} \quad \quad \quad \quad \quad \quad \\
\begin{array}{c}
\{2\} \\
1 \\
\end{array}
\]

- **Partially labelled graphs** are defined as usual, except that the node labelling function is a partial function.
- **Graph morphisms** between partially labelled graphs are defined as usual, except that unlabelled nodes can be mapped to arbitrary nodes.
- **Pushouts** are defined as usual, except that their graphs are partially labelled.
Graph transformation with relabelling

There is a price to pay:

1. Pushouts need not exist:

```
   2  -->  1
    ↓    ↓    ↓    ↓
    2  -->  ?
```
Graph transformation with relabelling

There is a price to pay:

1. Pushouts need not exist:

2. Pushout complements need not be unique:
Graph transformation with relabelling

Solution: *natural* pushouts

- A pushout is *natural* if it is also a pullback.
- A non-natural pushout:

  ![Diagram](attachment:pushout.png)

- Note: For totally labelled graphs, pushouts with injective morphisms are always natural.
Lemma (Habel-P 2002)

A pushout of partially labelled graphs

is natural if and only if for each unlabelled item \( x \) in \( A \), \( b(x) \) or \( c(x) \) is unlabelled.
Direct derivations with relabelling

- Consider rules $r = \langle L \leftarrow K \rightarrow R \rangle$ where $K$ is partially labelled and $L, R$ are totally labelled.
- Given an injective morphism $g : L \rightarrow G$, a *direct derivation* $G \Rightarrow_{r,g} H$ consists of two natural pushouts of the form

```
  L  K  R
/ \  / \  / \  / \  / \\  / \  / \  / \  / \\  / \  / \  / \  / \\  / \  / \  / \  / \\  / \  / \  / \  / \\ / \  / \  / \  / \
G  D  H
```

Proposition (Habel-P 2002)

*Given $r$ and $g$ as above, there exists a direct derivation $G \Rightarrow_{r,g} H$ if and only if $g$ satisfies the dangling condition. Moreover, in this case $D$ and $H$ are determined uniquely up to isomorphism.*
Proposition (Habel-P 2002)

Given a rule \( r = \langle L \leftarrow K \rightarrow R \rangle \) and an injective morphism \( g : L \rightarrow G \) satisfying the dangling condition, graphs \( D \) and \( H \) as above can be constructed as follows:

1. **Constructing \( D \) from \( G \):**
   - Remove all items in \( g(L) - g(K) \).
   - For each unlabelled node \( v \) in \( K \), make \( g_V(v) \) unlabelled.

2. **Constructing \( H \) from \( D \):**
   - Add disjointly all items from \( R - K \) while keeping their labels.
   - For \( e \in E_R - E_K \), \( s_H(e) \) is \( s_R(e) \) if \( s_R(e) \in V_R - V_K \), otherwise \( g_V(s_R(e)) \). Targets are defined analogously.
   - For each unlabelled node \( v \) in \( K \), label \( g_V(v) \) with \( l_R(v) \).
Example: direct derivation with relabelling
GP 2 host graph labels and type hierarchy

Label ::= List [Mark]
List ::= empty | Atom | List ‘:’ List
Atom ::= Integer | String
Integer ::= [‘-’] Digit {Digit}
String ::= ‘“’{Character}‘”’
Mark ::= red | green | blue | grey | dashed

\[
\text{list} \\
\text{atom} \\
\text{int} \quad \text{string} \\
\text{char}
\]

\[
(Z \cup \text{Char}^*)^* \\
\text{int} \quad \text{string} \\
\text{char}
\]
Example: GP 2 host graph
Rule schemata for attributed graph transformation

bridge($n$: int; $s, t$: string; $a$: atom; $x, y$: list)

where $n < 0$ and not edge(1, 3)
Rule schemata for attributed graph transformation

bridge($n: \text{int}; s, t: \text{string}; a: \text{atom}; x, y: \text{list}$)

where $n < 0$ and not edge($1, 3$)

- ‘:’ is list concatenation
- LHS expressions are simple (e.g. no operators except ‘:’ and ‘-’)
- Variables in RHS and condition must occur in LHS
Applying $\langle L \Rightarrow R, c \rangle$ to a host graph $G$:

1. Find injective premorphism $g : L \to G$ (ignoring labels)
2. Check if $g$ induces variable assignment $\alpha$ such that $g : L^\alpha \to G$ is label-preserving
3. Check whether $c^{\alpha,g} = \text{true}$
4. Apply rule instance $L^\alpha \Rightarrow R^{\alpha,g}$ with match $g$
Rule-schema application (sketched)

Applying $\langle L \Rightarrow R, c \rangle$ to a host graph $G$:

1. Find injective premorphism $g : L \rightarrow G$ (ignoring labels)
2. Check if $g$ induces variable assignment $\alpha$ such that $g : L^\alpha \rightarrow G$ is label-preserving
3. Check whether $c^{\alpha,g} = \text{true}$
4. Apply rule instance $L^\alpha \Rightarrow R^{\alpha,g}$ with match $g$

where $L^\alpha$, $R^{\alpha,g}$ and $c^{\alpha,g}$ result from

- replacing variables $x$ with $\alpha(x)$,
- replacing node identifiers $v$ with $g(v)$, and
- evaluating the resulting expressions.
Example: rule-schema application

\[ n \xrightarrow{s} a:x \xrightarrow{t} y \Rightarrow x:y \xrightarrow{s} 4 \xrightarrow{t} n*n \]

\[-5 \xrightarrow{"b"} 0:1:2 \xrightarrow{"c"} 3 \]

\[ 1 \xrightarrow{} 0 \xrightarrow{} 2 \]
Example: rule-schema application

\[
\begin{align*}
\text{n} & \xrightarrow{s} \text{a:x} & \xrightarrow{t} \text{y} & \Rightarrow & \text{x:y} & \xrightarrow{s} \text{4} & \xrightarrow{t} \text{n*n} \\
\downarrow g & & & & & & \\
\end{align*}
\]
Example: rule-schema application

\[
\begin{align*}
\text{n} & \quad \text{a:x} & \quad \text{y} \\
1 & \quad 2 & \quad 3 \\
\text{s} & \quad \implies & \quad \text{x:y} \\
\text{t} & \quad \downarrow g & \quad \text{n \times n} \\
\text{1} & \quad 4 & \quad 3 \\
\alpha \left\{ \begin{array}{l}
\text{n} \leftrightarrow -5 \\
\text{a} \leftrightarrow 0 \\
\text{x} \leftrightarrow 1:2 \\
\text{y} \leftrightarrow 3 \\
\text{s} \leftrightarrow "b" \\
\text{t} \leftrightarrow "c"
\end{array} \right. 
\end{align*}
\]
Example: rule-schema application

\[
\begin{align*}
\text{Example: rule-schema application} \\
\begin{array}{c}
\text{Example: rule-schema application}
\end{array}
\end{align*}
\]
Example: rule-schema application

\[ n \quad \frac{s}{a:x} \quad \frac{t}{y} \quad \Rightarrow \quad x:y \quad \frac{s}{4} \quad \frac{t}{n*n} \]

\[ \downarrow \alpha \]

\[ -5 \quad \frac{"b"}{0:1:2} \quad \frac{"c"}{3} \]

\[ \downarrow g \]

\[ \alpha \]

\[ \begin{align*}
  n & \mapsto -5 \\
  a & \mapsto 0 \\
  x & \mapsto 1:2 \\
  y & \mapsto 3 \\
  s & \mapsto "b" \\
  t & \mapsto "c" 
\end{align*} \]

\[ (n < 0 \text{ and not edge}(1, 3))_{\alpha, g} \]

\[ \equiv -5 < 0 \text{ and not edge}(1, 3) \]

\[ \equiv \text{true} \]
Example: rule-schema application

\[
\begin{align*}
\text{Example: rule-schema application} \\
\begin{array}{c}
\begin{tikzpicture}[node distance=1cm, every node/.style={circle,draw}]
  \node (n) {n}; 
  \node [right of=n] (a) {a:x}; 
  \node [right of=a] (y) {y}; 
\end{tikzpicture} \\
\Rightarrow \\
\begin{tikzpicture}[node distance=1cm, every node/.style={circle,draw}]
  \node (x:y) {x:y}; 
  \node [right of=x:y] (4) {4}; 
  \node [right of=4] (n*n) {n*n}; 
\end{tikzpicture}
\end{array}
\end{align*}
\]

\[
\downarrow \alpha
\]

\[
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}[node distance=1cm, every node/.style={circle,draw}]
  \node (n) {0:1:2}; 
  \node [right of=n] (3) {3}; 
\end{tikzpicture} \\
\Rightarrow \\
\begin{tikzpicture}[node distance=1cm, every node/.style={circle,draw}]
  \node (1:2:3) {1:2:3}; 
  \node [right of=1:2:3] (4) {4}; 
  \node [right of=4] (25) {25}; 
\end{tikzpicture}
\end{array}
\end{align*}
\]

\[
\downarrow g
\]

\[
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}[node distance=1cm, every node/.style={circle,draw}]
  \node (n) {0:1:2}; 
  \node [right of=n] (3) {3}; 
\end{tikzpicture} \\
\alpha \\
\begin{array}{ll}
  n & \mapsto -5 \\
  a & \mapsto 0 \\
  x & \mapsto 1:2 \\
  y & \mapsto 3 \\
  s & \mapsto "b" \\
  t & \mapsto "c"
\end{array}
\end{array}
\end{align*}
\]
Example: rule-schema application

\[ n \xrightarrow{s} a:x \xrightarrow{t} y \Rightarrow x:y \xrightarrow{s} 4 \xrightarrow{t} n \times n \]

\[ \downarrow \alpha \]

\[ \downarrow \alpha, g \]

\[ \downarrow g \]

\[ \downarrow h \]
Part III

Graph Programs
Syntax of rule-schema labels

Label ::= ListExp [Mark]
ListExp ::= empty | ListVar | AtomExp
           | ListExp ':' ListExp
AtomExp ::= AtomVar | IntExp | StringExp
IntExp ::= IntVar | Number
           | (indeg | outdeg) '(' NodId ')'
           | length '(' (AtomVar | StringVar | ListVar) ')
           | '-' IntExp
           | '(' IntExp ')'
           | IntExp ('+' | '-' | '*' | '/') IntExp
StringExp ::= StringVar | CharVar | ' " ' {Character} ' " ' 
            | StringExp '.' StringExp
Mark ::= red | green | blue | grey | dashed | any
Syntax of rule-schema conditions

**Condition** ::= edge ‘(’ NodeID ‘,’ NodeID [ ‘,’ Label ] ‘)’
| SubtypeExp
| ListExp ( ‘=’ | ‘!=’ ) ListExp
| IntExp ( ‘>’ | ‘>=’ | ‘<’ | ‘<=' ) IntExp
| not Condition
| Condition ( and | or ) Condition
| ‘(’ Condition ‘)’

**SubtypeExp** ::= (atom | int | string | char) ‘(’ Var ‘)’
Syntax of commands

Program ::= Decl {Decl}
Decl ::= RuleDecl | ProcDecl | MainDecl
ProcDecl ::= ProcId ‘=’ [LocalDecl] ComSeq
MainDecl ::= Main ‘=’ ComSeq
ComSeq ::= Com {‘;’ Com}
Com ::= RuleSetCall | ProcCall
      | if ComSeq then ComSeq [else ComSeq]
      | try ComSeq [then ComSeq [else ComSeq]]
      | ComSeq ‘!’
      | ComSeq or ComSeq
      | ‘(’ ComSeq ‘)’
      | break | skip | fail
RuleSetCall ::= RuleId ‘{’ [RuleId {‘,’ RuleId}] ‘}’
ProcCall ::= ProcId
Example program: transitive closure

Main = link!

link(a, b, x, y, z: list)

where not edge(1, 3)
Proposition (Termination)

On every input graph $G$, the program terminates after at most $|V_G|^2$ rule schema applications.

Proof

Given any graph $X$, let

$$\#X = |\{\langle v, w \rangle \mid v, w \in V_X \text{ and there is no edge } v \rightarrow w\}|.$$  

Note that $\#X \leq |V_X|^2$. Moreover, for every step $G \Rightarrow_{\text{link}} H$, $\#H = \#G - 1$. Hence $\text{link!}$ terminates after at most $|V_G|^2$ rule schema applications.  

$\square$
Example program: transitive closure (cont’d)

Proposition (Correctness)

The program returns a transitive closure of the input graph.

Proof

Let $G$ be the input graph and $T$ the resulting graph. For every step $X \Rightarrow_{\text{link}} Y$, there is an injective graph morphism $X \to Y$ because $\text{link}$ does not delete or relabel any items. It follows that $T$ is an extension of $G$ (up to isomorphism).

We show that $T$ is transitive by induction on the length of paths in $T$. Consider a directed path $v_0, v_1, \ldots, v_n$ with $v_0 \neq v_n$. We can assume w.l.o.g. that $v_0, \ldots, v_n$ are distinct. If $n = 1$, there is an edge $v_0 \to v_n$. If $n > 1$, there is an edge $v_0 \to v_{n-1}$ by induction hypothesis. Thus there are edges $v_0 \to v_{n-1} \to v_n$. As $\text{link}$ has been applied as long as possible, there must be an edge $v_0 \to v_n$. Finally, $T$ is a smallest transitive extension of $G$ because whenever $\text{link}$ creates an edge $v \to v'$, by the declaration of $\text{link}$ there is no such edge but a path $v \leadsto v'$. □
Example program: graph inverse

The **inverse** of a graph is obtained by reversing all edges.

\[
\text{Main} = \text{reverse!;} \; \text{unmark!}
\]

\[
\text{reverse}(a,x,y: \text{list})
\]

\[
\begin{array}{c}
\text{x} \quad \text{a} \quad \text{y} \\
1 \quad \quad 2
\end{array}
\Rightarrow
\begin{array}{c}
\text{x} \quad \quad \text{a} \quad \text{y} \\
1 \quad \quad 2
\end{array}
\]

\[
\text{unmark}(a,x,y: \text{int})
\]

\[
\begin{array}{c}
\text{x} \quad \quad \text{a} \quad \text{y} \\
1 \quad \quad 2
\end{array}
\Rightarrow
\begin{array}{c}
\text{x} \quad \quad \text{a} \\
1 \quad \quad 2
\end{array}
\]
A *vertex colouring* is an assignment of colours to nodes such that each non-loop edge has end points with distinct colours.

Main = mark!; init!; inc!

mark(x: list)

\[
\begin{array}{c}
\text{x} \\
1
\end{array} \quad \Rightarrow \quad \\
\begin{array}{c}
\text{x} \\
1
\end{array}
\]

init(x: list)

\[
\begin{array}{c}
\text{x} \\
1
\end{array} \quad \Rightarrow \quad \\
\begin{array}{c}
\text{x:1} \\
1
\end{array}
\]

inc(a, x, y: list; i: int)

\[
\begin{array}{c}
x:i \\
1
\end{array} \quad \xrightarrow{a} \quad \\
\begin{array}{c}
y:i \\
2
\end{array} \quad \Rightarrow \quad \\
\begin{array}{c}
x:i \\
1
\end{array} \quad \xrightarrow{a} \quad \\
\begin{array}{c}
y:i+1 \\
2
\end{array}
\]
Example program: vertex colouring (cont’d)
Partial correctness of vertex colouring

A graph is *correctly coloured* if the end points of each non-loop edge have labels of the form \( x:i \) and \( y:j \), for integers \( i, j \) with \( i \neq j \)

**Proposition (Partial correctness)**

*If the program terminates on an input graph \( G \), then it returns \( G \) correctly coloured.*

**Proof**

Consider a terminating program run

\[
G \xrightarrow{\text{mark}} G' \xrightarrow{\text{init}} H \xrightarrow{\text{inc}} M.
\]

Then \( H \) is obtained from \( G \) by replacing each node label \( x \) with \( x:1 \). Suppose that \( M \) is not correctly coloured. Then it contains a non-loop edge whose end points have the same colour. Hence \( \text{inc} \) is applicable to \( M \), contradicting the fact that \( M \) results from applying \( \text{inc} \) as long as possible.

\[\square\]
Termination of vertex colouring

For a coloured graph \( G \), let \( \text{Colours}(G) = \{\text{colour}(v) \mid v \in V_G\} \)
where for any node \( v \) with label \( x:i \), \( \text{colour}(v) = i \)

**Lemma (Invariant)**

*Given any derivation* \( G \Rightarrow^*_\text{inc} H \) *with* \( \text{Colours}(G) = \{1\} \),

\[
\text{Colours}(H) = \{i \mid 1 \leq i \leq n\} \text{ for some } 1 \leq n \leq |V_H|.
\]

**Proof**

Every step \( X \Rightarrow^*_\text{inc} Y \) satisfies \( \text{Colours}(Y) = \text{Colours}(X) \cup \Delta \),
where \( \Delta = \emptyset \) or \( \Delta = \{\max(\text{Colours}(X)) + 1\} \). The invariant then follows by induction on the length of \( G \Rightarrow^*_\text{inc} H \). □
Termination of vertex colouring (cont’d)

Proposition (Termination)

On every input graph $G$, the program terminates after $O(|V_G|^2)$ rule applications.

Proof

Both mark! and init! terminate after $|V_G|$ steps.

Let $G$ be a coloured host graph with $\text{Colours}(G) = \{1\}$ and suppose there was an infinite derivation

$$G = G_0 \Rightarrow_{\text{inc}} G_1 \Rightarrow_{\text{inc}} G_2 \Rightarrow_{\text{inc}} \ldots$$

For $i \geq 0$, let $\#G_i = \sum_{v \in V_{G_i}} \text{colour}(v)$. Then

$$\#G_i < \#G_{i+1} \text{ for every } i \geq 0$$

by the labelling of inc. (continued)
Termination of vertex colouring (cont’d)

But the invariant shows that for all \( i \geq 0 \),

\[
\#G_i \leq \sum_{j=1}^{\#V_{G_i}} j = \sum_{j=1}^{\#V_G} j
\]

where \( \#V_{G_i} = \#V_G \) because \( \text{inc} \) preserves the number of nodes. Thus the infinite sequence \( G = G_0 \Rightarrow_{\text{inc}} G_1 \Rightarrow_{\text{inc}} \ldots \) cannot exist.

Also, any sequence of \( \text{inc} \) applications starting from \( G \) has at most a quadratic length because

\[
\sum_{j=1}^{\#V_G} j = \frac{\#V_G \times (\#V_G + 1)}{2}.
\]
Example program: 2-colouring

Main = try(mark!; init; colour!; if illegal then fail)

mark(x: list)

\[
\begin{array}{c}
\text{x} \\
1
\end{array} 
\Rightarrow
\begin{array}{c}
\text{x} \\
1
\end{array}
\]

init(x: list)

\[
\begin{array}{c}
\text{x} \\
1
\end{array} 
\Rightarrow
\begin{array}{c}
\text{x}:1 \\
1
\end{array}
\]

colour(a, x, y: list; i: int)

\[
\begin{array}{c}
\text{x}:i \\
1
\end{array}
\quad \text{a}
\quad \begin{array}{c}
\text{y} \\
2
\end{array} 
\Rightarrow
\begin{array}{c}
\text{x}:i \\
1
\end{array}
\quad \text{a}
\quad \begin{array}{c}
\text{y}:1-i \\
2
\end{array}
\]

illegal(a, x, y: list; i: int)

\[
\begin{array}{c}
\text{x}:i \\
1
\end{array}
\quad \text{a}
\quad \begin{array}{c}
\text{y}:i \\
2
\end{array} 
\Rightarrow
\begin{array}{c}
\text{x}:i \\
1
\end{array}
\quad \text{a}
\quad \begin{array}{c}
\text{y}:i \\
2
\end{array}
\]
Example program: 2-colouring

Main = try(mark!; init; colour!; if illegal then fail)

mark(x: list)

init(x: list)

colour(a, x, y: list; i: int)

illegal(a, x, y: list; i: int)

Assumption: input graph is connected
Example program: 2-colouring (cont’d)

\[\begin{array}{c}
\begin{array}{c}
\text{Node 1} \\
\text{Node 2} \\
\text{Node 3} \\
\text{Node 4}
\end{array}
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
\text{Node 0} \\
\text{Node 1}
\end{array}
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
\text{Node 0} \\
\text{Node 1}
\end{array}
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
\text{Node 0} \\
\text{Node 1}
\end{array}
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
\text{Node 0} \\
\text{Node 1}
\end{array}
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
\text{Node 0} \\
\text{Node 1}
\end{array}
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
\text{Node 0} \\
\text{Node 1}
\end{array}
\end{array}\]
Lemma (2-colourability)

A graph is 2-colourable if and only if it does not contain an undirected cycle of odd length $\geq 3$.

Proof

See any textbook on graph theory.

Proposition (Termination)

On every input graph $G$, the program terminates after $O(\vert V_G \vert)$ rule schema applications.

Proposition (Correctness)

Given a connected input graph $G$, the program either returns $G$ 2-coloured or, if $G$ is not 2-colourable, returns $G$ unmodified.
Example: 2-colouring for disconnected graphs (Version 1)

\[
\text{Main} = \text{try} (\text{mark}!; (\text{init}; \text{colour})!; \text{if illegal then fail})
\]

\[
\text{mark}(x: \text{list})
\]

\[
\begin{array}{c}
\text{x} \\
1
\end{array}
\Rightarrow
\begin{array}{c}
\text{x} \\
1
\end{array}
\]

\[
\text{init}(x: \text{list})
\]

\[
\begin{array}{c}
\text{x} \\
1
\end{array}
\Rightarrow
\begin{array}{c}
x:1 \\
1
\end{array}
\]

\[
\text{colour}(a, x, y: \text{list}; i: \text{int})
\]

\[
\begin{array}{c}
x:i \\
1
\end{array}
\text{a}
\begin{array}{c}
y \\
2
\end{array}
\Rightarrow
\begin{array}{c}
x:i \\
1
\end{array}
\text{a}
\begin{array}{c}
y:1-i \\
2
\end{array}
\]

\[
\text{illegal}(a, x, y: \text{list}; i: \text{int})
\]

\[
\begin{array}{c}
x:i \\
1
\end{array}
\text{a}
\begin{array}{c}
y:i \\
2
\end{array}
\Rightarrow
\begin{array}{c}
x:i \\
1
\end{array}
\text{a}
\begin{array}{c}
y:i \\
2
\end{array}
\]
Example: 2-colouring for disconnected graphs (Version 2)

Main = (init; colour)!: if illegal then undo! else unmark!

init(x: list)

\[
\begin{align*}
x & \Rightarrow x:1 \\
1 & \Rightarrow 1 \\
\end{align*}
\]

colour(a, x, y: list; i: int)

\[
\begin{align*}
x:i & \Rightarrow x:i \\
1 & \Rightarrow 1 \\
\end{align*}
\]

illegal(a, x, y: list; i: int)

\[
\begin{align*}
x:i & \Rightarrow x:i \\
1 & \Rightarrow 1 \\
\end{align*}
\]

undo(x: list; i: int)

\[
\begin{align*}
x:i & \Rightarrow x \\
1 & \Rightarrow 1 \\
\end{align*}
\]

unmark(x: list)

\[
\begin{align*}
x & \Rightarrow x \\
1 & \Rightarrow 1 \\
\end{align*}
\]
Example program: shortest distances

Main = init; {add, reduce}!

init(x: list)

\[
\begin{array}{c}
\text{x} \quad \Rightarrow \\
\text{x:0}
\end{array}
\]

add(x, y: list; m, n: int)

\[
\begin{array}{c}
x:m \\ 1
\end{array}
\quad \Rightarrow \\
\begin{array}{c}
y \\ 2
\end{array}
\quad \Rightarrow \\
\begin{array}{c}
x:m \\ 1
\end{array}
\quad \Rightarrow \\
\begin{array}{c}
y:m+n \\ 2
\end{array}
\]

reduce(x, y: list; m, n, p: int)

\[
\begin{array}{c}
x:m \\ 1
\end{array}
\quad \Rightarrow \\
\begin{array}{c}
y:p \\ 2
\end{array}
\quad \Rightarrow \\
\begin{array}{c}
x:m \\ 1
\end{array}
\quad \Rightarrow \\
\begin{array}{c}
y:m+n \\ 2
\end{array}
\]

where \( m + n < p \)
Example program: shortest distances

Main = init; {add, reduce}!

init(x: list)

\[
\begin{array}{c}
x_1 \\ \Rightarrow \\
\end{array} \quad \begin{array}{c}
x:0_1 \\ 
\end{array}
\]

add(x, y: list; m, n: int)

\[
\begin{array}{c}
x:m_1 \quad \text{\(\quad\)} \quad y_{2} \\ \Rightarrow \\
x:m_1 \\ \quad \text{\(\quad\)} \quad y:m+n_{2}
\end{array}
\]

reduce(x, y: list; m, n, p: int)

\[
\begin{array}{c}
x:m_1 \\ \quad \text{\(\quad\)} \quad y:p_2 \\ \Rightarrow \\
x:m_1 \\ \quad \text{\(\quad\)} \quad y:m+n_{2}
\end{array}
\]

where \(m + n < p\)

Assumption: input graph has a single red node; edge labels are non-negative integers
Proposition (Termination)

The program terminates on every input graph whose edge labels are non-negative integers.

Proof

Suppose that the program does not terminate on some input graph. Then there is an infinite sequence $G_0 \Rightarrow r_0 \ G_1 \Rightarrow r_1 \ G_2 \Rightarrow \ldots$ with $r_i \in \{\text{add}, \text{reduce}\}$ for all $i \geq 0$. Because add decreases the number of unmarked nodes, and reduce preserves this number, there must be some $k \geq 0$ such that $r_i = \text{reduce}$ for all $i \geq k$.

Given a graph $G_i$ in the sequence, let $\#G_i$ be the sum of all distances in $G_i$'s marked nodes. Then $\#G_i > \#G_{i+1}$ for each step $G_i \Rightarrow \text{reduce} \ G_{i+1}$. Since all edge labels in $G_0$ are non-negative, all distances in the marked graphs of the sequence are non-negative (an invariant easily proved by induction). But this contradicts the infinity of $G_k \Rightarrow G_{k+1} \Rightarrow G_{k+2} \Rightarrow \ldots$
Lemma (Invariant)

If $G \Rightarrow_{\text{init}} G' \Rightarrow^{*}_{\{\text{add, reduce}\}} H$, then each marked node $v$ in $H$ has a label $x : n$ where $n$ is a distance from the red node to $v$.

Proof

By induction on the length of $G' \Rightarrow^{*}_{\{\text{add, reduce}\}} H$. 

Proposition (Correctness)

Consider an input graph whose edge labels are non-negative integers and suppose that there is a single red node $v$. Then the program marks each node $w$ that is reachable from $v$ and adds to $w$'s label the shortest distance from $v$ to $w$. 
Equivalent program for shortest distances

Main = init; add!; reduce!

init(x: list)

\[
x \quad \Rightarrow \quad x:0
\]

add(x, y: list; m, n: int)

\[
x:m \quad \Rightarrow \quad x:m \quad y \quad \Rightarrow \quad x:m \quad y:m+n
\]

reduce(x, y: list; m, n, p: int)

\[
x:m \quad y:p \quad \Rightarrow \quad x:m \quad y:m+n
\] where \( m + n < p \)

Note: \{add, reduce\} can be shown to be confluent by a (non-trivial) critical-pair analysis [Hristakiev 17]
Example program: recognising cyclic graphs

Main = if Cyclic then \( P \) else \( Q \)
Cyclic = delete!; \{edge, loop\}

\[
delete(a, x, y: list)\]
\[
\begin{array}{c}
\text{x} \\
1
\end{array} \quad \begin{array}{c}
\text{a} \\
\end{array} \quad \begin{array}{c}
\text{y} \\
2
\end{array} \\
\Rightarrow \quad \begin{array}{c}
\text{x} \\
1
\end{array} \quad \begin{array}{c}
\text{y} \\
2
\end{array}
\]

\[
\text{where indeg}(1) = 0
\]

\[
\text{edge}(a, x, y: list)\]
\[
\begin{array}{c}
\text{x} \\
1
\end{array} \quad \begin{array}{c}
\text{a} \\
\end{array} \quad \begin{array}{c}
\text{y} \\
2
\end{array} \\
\Rightarrow \quad \begin{array}{c}
\text{x} \\
1
\end{array} \quad \begin{array}{c}
\text{a} \\
\end{array} \quad \begin{array}{c}
\text{y} \\
2
\end{array}
\]

\[
\text{loop}(a, x: list)\]
\[
\begin{array}{c}
\text{x} \\
1
\end{array} \quad \begin{array}{c}
\text{a} \\
\end{array} \\
\Rightarrow \quad \begin{array}{c}
\text{x} \\
1
\end{array} \quad \begin{array}{c}
\text{a} \\
\end{array}
\]

\[
/* \text{preserves cycles} \]
\[
\text{and cycle-freeness} */
\]
Example program: recognising cyclic graphs (cont’d)

$G:\quad \rightarrow \quad \rightarrow \quad \rightarrow \quad \rightarrow$

$H:\quad \rightarrow \quad \rightarrow \quad \rightarrow \quad \rightarrow$

- $\downarrow^*$ succeeds, $P$ is executed on $G$
- $\downarrow^*$ succeeds, $Q$ is executed on $H$

- $\{\text{edge, loop}\}$ fails, $Q$ is executed on $H$
Series-parallel graphs are inductively defined:

1. $s \rightarrow t$ is series-parallel, where $s$ is the source and $t$ is the sink.
2. The class of series-parallel graphs is closed under parallel composition and sequential composition.

(Wikipedia)
Example program: recognising series-parallel graphs

**Series-parallel graphs** are inductively defined:

1. $s \xrightarrow{} t$ is series-parallel, where $s$ is the source and $t$ is the sink.

2. The class of series-parallel graphs is closed under *parallel composition* and *sequential composition*.

Equivalently, a graph is series-parallel if it can be reduced to $s \xrightarrow{} t$ by the following rules:

$$
\begin{align*}
&1 \xrightarrow{} 0 \xrightarrow{} 0 \Rightarrow 1 \xrightarrow{} 0 \\
&1 \xrightarrow{} 0 \xrightarrow{} 0 \Rightarrow 1 \xrightarrow{} 0
\end{align*}
$$

(Wikipedia)
Example program: recognising series-parallel graphs

Series-parallel = Reduce!; delete; if nonempty then fail
Reduce = \{series, parallel\}

\textbf{series}(a, b, x, y, z: list)

\begin{tikzpicture}[node distance=1.5cm,thick]
  \node (x) {x};
  \node (y) [right of=x] {y};
  \node (z) [right of=y] {z};
  \path (x) edge node {a} (y);
  \path (y) edge node {b} (z);
  \node (x1) at (0,0) {1};
  \node (y1) at (2,0) {2};
  \node (z1) at (4,0) {1};
  \node (z2) at (4,2) {2};

  \node (x2) at (6,0) {x};
  \node (y2) [right of=x2] {y};
  \node (z2) [right of=y2] {z};
  \node (x12) at (8,0) {1};
  \node (y12) at (10,0) {2};

  \path (x2) edge node {} (y2);
  \path (y2) edge node {} (z2);
  \path (x12) edge node {} (y12);

  \draw[->] (y) -- (z);
  \draw[->] (x) -- (y);
  \draw[->] (y) -- (z);

  \draw[->] (x1) -- (y1);
  \draw[->] (y1) -- (z1);
  \draw[->] (x12) -- (y12);

  \node (x3) at (12,0) {x};
  \node (y3) [right of=x3] {y};
  \node (z3) [right of=y3] {z};
  \node (x13) at (14,0) {1};
  \node (y13) at (16,0) {2};

  \path (x3) edge node {} (y3);
  \path (y3) edge node {} (z3);
  \path (x13) edge node {} (y13);

  \draw[->] (y3) -- (z3);
  \draw[->] (x3) -- (y3);
  \draw[->] (y3) -- (z3);

  \draw[->] (x13) -- (y13);
  \draw[->] (y13) -- (z13);

  \end{tikzpicture}

\textbf{parallel}(a, b, x, y: list)

\begin{tikzpicture}[node distance=1.5cm,thick]
  \node (x) {x};
  \node (y) [right of=x] {y};
  \path (x) edge node {a} (y);
  \node (x1) at (0,0) {1};
  \node (y1) at (2,0) {2};
  \node (x2) at (4,0) {1};

  \node (x22) at (6,0) {x};
  \node (y22) [right of=x22] {y};
  \node (x122) at (8,0) {1};

  \path (x22) edge node {} (y22);
  \path (x22) edge node {} (y22);
  \path (x122) edge node {} (y122);

  \draw[->] (x) -- (y);
  \draw[->] (x) -- (y);
  \draw[->] (x) -- (y);

  \draw[->] (x1) -- (y1);
  \draw[->] (x1) -- (y1);
  \draw[->] (x1) -- (y1);

  \draw[->] (x12) -- (y12);
  \draw[->] (x12) -- (y12);

  \end{tikzpicture}

\textbf{delete}(a, x, y: list) \quad \textbf{nonempty}(x: list)

\begin{tikzpicture}[node distance=1.5cm,thick]
  \node (x) {x};
  \node (y) [right of=x] {y};
  \path (x) edge node {a} (y);
  \node (x1) at (0,0) {1};
  \node (y1) at (2,0) {2};

  \node (x2) at (4,0) {x};
  \node (y2) [right of=x2] {y};
  \node (x12) at (6,0) {1};

  \path (x2) edge node {} (y2);
  \path (x2) edge node {} (y2);
  \path (x12) edge node {} (y12);

  \draw[->] (x) -- (y);
  \draw[->] (x) -- (y);

  \draw[->] (x1) -- (y1);
  \draw[->] (x1) -- (y1);

  \draw[->] (x12) -- (y12);
  \draw[->] (x12) -- (y12);

  \end{tikzpicture}

\begin{tikzpicture}[node distance=1.5cm,thick]
  \node (x) {x};
  \node (y) [right of=x] {y};
  \path (x) edge node {a} (y);
  \node (x1) at (0,0) {1};

  \node (x2) at (4,0) {x};
  \node (y2) [right of=x2] {y};
  \node (x12) at (6,0) {1};

  \path (x2) edge node {} (y2);
  \path (x2) edge node {} (y2);
  \path (x12) edge node {} (y12);

  \draw[->] (x) -- (y);
  \draw[->] (x) -- (y);

  \draw[->] (x1) -- (y1);

  \draw[->] (x12) -- (y12);
  \draw[->] (x12) -- (y12);

  \end{tikzpicture}

\begin{tikzpicture}[node distance=1.5cm,thick]
  \node (x) {x};
  \node (y) [right of=x] {y};
  \path (x) edge node {a} (y);
  \node (x1) at (0,0) {1};

  \node (x2) at (4,0) {x};
  \node (y2) [right of=x2] {y};
  \node (x12) at (6,0) {1};

  \path (x2) edge node {} (y2);
  \path (x2) edge node {} (y2);
  \path (x12) edge node {} (y12);

  \draw[->] (x) -- (y);
  \draw[->] (x) -- (y);

  \draw[->] (x1) -- (y1);

  \draw[->] (x12) -- (y12);
  \draw[->] (x12) -- (y12);

  \end{tikzpicture}

\begin{tikzpicture}[node distance=1.5cm,thick]
  \node (x) {x};
  \node (y) [right of=x] {y};
  \path (x) edge node {a} (y);
  \node (x1) at (0,0) {1};

  \node (x2) at (4,0) {x};
  \node (y2) [right of=x2] {y};
  \node (x12) at (6,0) {1};

  \path (x2) edge node {} (y2);
  \path (x2) edge node {} (y2);
  \path (x12) edge node {} (y12);

  \draw[->] (x) -- (y);
  \draw[->] (x) -- (y);

  \draw[->] (x1) -- (y1);

  \draw[->] (x12) -- (y12);
  \draw[->] (x12) -- (y12);

  \end{tikzpicture}
Termination of Series-parallel

Proposition (Termination)

On every input graph $G$, Series-parallel terminates after at most $|E_G| + 1$ rule schema applications.

Proof

The maximum number of rule schema applications is given if Reduce! deletes all but one edges, delete removes the last edge, and nonempty is applicable to some remaining node. 

\[\square\]
Correctness of Series-parallel

Proposition (Correctness)

Given any input graph $G$, Series-parallel returns the empty graph if $G$ is series-parallel, and fails otherwise.

Proof

If $G$ is series-parallel, every execution of Reduce! on $G$ results in the graph TwoNodes consisting of two nodes and one non-loop edge. This is because Reduce is confluent on series-parallel graphs:

$$ \text{TwoNodes} \xleftarrow{*} \text{Reduce} G \Rightarrow \text{Reduce} H $$

implies $H \Rightarrow^*_{\text{Reduce}} \text{TwoNodes}$. This can be shown by critical-pair analysis [Hristakiev 17].

The application of delete to TwoNodes will then return $\emptyset$ and the if-command has no effect.

If $G$ is not series-parallel, either delete fails or it is applicable but produces a non-empty graph on which the if-command fails. □
Part IV

Operational Semantics
Structural operational semantics (SOS)

- **Configurations** are given by \((\text{ComSeq} \times \mathcal{G}) \cup \mathcal{G} \cup \{\text{fail}\}\), where \(\mathcal{G}\) consists of all host graphs

- **Results** are configurations in \(\mathcal{G} \cup \{\text{fail}\}\)

- **Transition relation**

  \[ \rightarrow \subseteq (\text{ComSeq} \times \mathcal{G}) \times ((\text{ComSeq} \times \mathcal{G}) \cup \mathcal{G} \cup \{\text{fail}\}) \]

  is defined by structural induction on command sequences
SOS: rule set-call $\mathcal{R}$ (set of rule schemata)

$[\text{call}_1] \quad \frac{G \Rightarrow_\mathcal{R} H}{\langle \mathcal{R}, G \rangle \rightarrow H}$

$[\text{call}_2] \quad \frac{G \not\Rightarrow_\mathcal{R}}{\langle \mathcal{R}, G \rangle \rightarrow \text{fail}}$
SOS: sequential composition

[seq\textsubscript{1}]
\[\langle P, G \rangle \rightarrow \langle P', H \rangle\]
\[\langle P; Q, G \rangle \rightarrow \langle P'; Q, H \rangle\]

[seq\textsubscript{2}]
\[\langle P, G \rangle \rightarrow H\]
\[\langle P; Q, G \rangle \rightarrow \langle Q, H \rangle\]

[seq\textsubscript{3}]
\[\langle P, G \rangle \rightarrow \text{fail}\]
\[\langle P; Q, G \rangle \rightarrow \text{fail}\]
Example: transition relation

Main = \{r_1, r_2\}; \{r_1, r_2\}; r_1!

\begin{align*}
\text{r}_1 & : 1 \Rightarrow 1 & \text{r}_2 & : 1 \Rightarrow 2 \\
\end{align*}

\[
\langle \text{Main}, 1 \rangle \rightarrow \langle P, 2 \rangle \rightarrow \text{fail} \downarrow \\
\langle P, 1 \rangle \rightarrow \langle \text{r}_1!, 1 \rangle \rightarrow \langle \text{r}_1!, 1 \rangle \rightarrow \ldots \\
\downarrow \\
\langle \text{r}_1!, 2 \rangle \downarrow \\
2
\]

where \( P = \{r_1, r_2\}; r_1! \)
SOS: if-then-else and try-then-else

[if\(_1\)] \[
\langle C, G \rangle \rightarrow^+ H
\]
\[\langle \text{if } C \text{ then } P \text{ else } Q, G \rangle \rightarrow \langle P, G \rangle\]

[if\(_2\)] \[
\langle C, G \rangle \rightarrow^+ \text{ fail}
\]
\[\langle \text{if } C \text{ then } P \text{ else } Q, G \rangle \rightarrow \langle Q, G \rangle\]

[try\(_1\)] \[
\langle C, G \rangle \rightarrow^+ H
\]
\[\langle \text{try } C \text{ then } P \text{ else } Q, G \rangle \rightarrow \langle P, H \rangle\]

[try\(_2\)] \[
\langle C, G \rangle \rightarrow^+ \text{ fail}
\]
\[\langle \text{try } C \text{ then } P \text{ else } Q, G \rangle \rightarrow \langle Q, G \rangle\]
SOS: as-long-as-possible iteration

[alap\textsubscript{1}] \quad \frac{\langle P, G \rangle \rightarrow^+ H}{\langle P!, G \rangle \rightarrow \langle P!, H \rangle}

[alap\textsubscript{2}] \quad \frac{\langle P, G \rangle \rightarrow^+ \text{fail}}{\langle P!, G \rangle \rightarrow G}
SOS: as-long-as-possible iteration

\[ \text{[alap}_1\text{]} \quad \frac{\langle P, G \rangle \rightarrow^+ H}{\langle P!, G \rangle \rightarrow \langle P!, H \rangle} \]

\[ \text{[alap}_2\text{]} \quad \frac{\langle P, G \rangle \rightarrow^+ \text{fail}}{\langle P!, G \rangle \rightarrow G} \]

Note: \( P! \) may be a nested loop
SOS: break

[break] \[ \frac{\langle \text{break}; P, G \rangle}{\langle \text{break}, G \rangle} \]

[alap\textsubscript{3}] \[ \frac{\langle P, G \rangle \rightarrow^* \langle \text{break}, H \rangle}{\langle P!, G \rangle \rightarrow H} \]
SOS: derived commands

\[\text{or}_1\] \( \langle P \text{ or } Q, G \rangle \rightarrow \langle P, G \rangle \)

\[\text{skip}\] \( \langle \text{skip}, G \rangle \rightarrow G \)

\[\text{if}_3\] \( \frac{\langle C, G \rangle \rightarrow^+ H}{\langle \text{if } C \text{ then } P, G \rangle \rightarrow \langle P, G \rangle} \)

\[\text{try}_3\] \( \frac{\langle C, G \rangle \rightarrow^+ H}{\langle \text{try } C \text{ then } P, G \rangle \rightarrow \langle P, H \rangle} \)

\[\text{or}_2\] \( \langle P \text{ or } Q, G \rangle \rightarrow \langle Q, G \rangle \)

\[\text{fail}\] \( \langle \text{fail}, G \rangle \rightarrow \text{fail} \)

\[\text{if}_4\] \( \frac{\langle C, G \rangle \rightarrow^+ \text{fail}}{\langle \text{if } C \text{ then } P, G \rangle \rightarrow G} \)

\[\text{try}_4\] \( \frac{\langle C, G \rangle \rightarrow^+ \text{fail}}{\langle \text{try } C \text{ then } P, G \rangle \rightarrow G} \)

\[\text{try}_5\] \( \frac{\langle C, G \rangle \rightarrow^+ H}{\langle \text{try } C, G \rangle \rightarrow H} \)

\[\text{try}_6\] \( \frac{\langle C, G \rangle \rightarrow^+ \text{fail}}{\langle \text{try } C, G \rangle \rightarrow G} \)
Semantic function $[\cdot]$:

$[\cdot]: \text{ComSeq} \to (\mathcal{G} \to 2^{\mathcal{G} \cup \{\text{fail}, \bot\}})$ is defined by

$$[P] G = \{X \in \mathcal{G} \cup \{\text{fail}\} | \langle P, G \rangle \xrightarrow{\top} X\} \cup \{ot | P \text{ can diverge or get stuck from } G\}$$

where

- $P$ can diverge from $G$ if there is an infinite sequence $\langle P, G \rangle \to \langle P_1, G_1 \rangle \to \langle P_2, G_2 \rangle \to \ldots$

- $P$ can get stuck from $G$ if there is a terminal configuration $\langle Q, H \rangle$ such that $\langle P, G \rangle \xrightarrow{*} \langle Q, H \rangle$ (where $Q$ cannot be executed because no inference rule is applicable)
Semantic equivalence

Programs $P$ and $Q$ are *semantically equivalent* if $\llbracket P \rrbracket = \llbracket Q \rrbracket$

Examples
Semantic equivalence

Programs $P$ and $Q$ are *semantically equivalent* if $\llbracket P \rrbracket = \llbracket Q \rrbracket$

Examples

- $\text{skip} \equiv \text{null}$
  - with null: $\emptyset \Rightarrow \emptyset$
Semantic equivalence

Programs $P$ and $Q$ are *semantically equivalent* if $\llbracket P \rrbracket = \llbracket Q \rrbracket$

Examples

- $\text{skip} \equiv \text{null}$
  with $\text{null}: \emptyset \Rightarrow \emptyset$

- $\text{fail} \equiv \{\}$
Semantic equivalence

Programs $P$ and $Q$ are *semantically equivalent* if $\llbracket P \rrbracket = \llbracket Q \rrbracket$

Examples

- $\text{skip} \equiv \text{null}$
  
  with null: $\emptyset \Rightarrow \emptyset$

- $\text{fail} \equiv \{\}$

- $\text{if } C \text{ then } P \equiv \text{if } C \text{ then } P \text{ else null}$
Semantic equivalence

Programs $P$ and $Q$ are *semantically equivalent* if $\llbracket P \rrbracket = \llbracket Q \rrbracket$

Examples

- skip $\equiv$ null
  with null: $\emptyset \Rightarrow \emptyset$

- fail $\equiv \{\}$

- if $C$ then $P$ $\equiv$ if $C$ then $P$ else null

- $P$ or $Q$ $\equiv$ if (Delete!; \{create, null\}; node) then $P$ else $Q$
  with Delete removing edges and isolated nodes, create: $\emptyset \Rightarrow \bigcirc$ and
  node: $\bigcirc \Rightarrow \bigcirc$


Semantic equivalence

Programs $P$ and $Q$ are semantically equivalent if $[P] = [Q]$

Examples

- skip $\equiv$ null
  with null: $\emptyset \Rightarrow \emptyset$
- fail $\equiv \{\}$
- if $C$ then $P \equiv$ if $C$ then $P$ else null
- $P$ or $Q \equiv$ if (Delete!; \{create, null\}; node) then $P$ else $Q$
  with Delete removing edges and isolated nodes, create: $\emptyset \Rightarrow \circ$ and node: $\circ \Rightarrow \circ$
- try $C$ then $P$ else $Q \not\equiv$ if $C$ then $(C; P)$ else $Q$
Semantic equivalence

Programs $P$ and $Q$ are *semantically equivalent* if $\llbracket P \rrbracket = \llbracket Q \rrbracket$

**Examples**

- skip $\equiv$ null
  with null: $\emptyset \Rightarrow \emptyset$

- fail $\equiv \{\}$

- if $C$ then $P \equiv$ if $C$ then $P$ else null

- $P$ or $Q \equiv$ if (Delete!; \{create, null\}; node) then $P$ else $Q$
  with Delete removing edges and isolated nodes, create: $\emptyset \Rightarrow \bigcirc$ and node: $\bigcirc \Rightarrow \bigcirc$

- try $C$ then $P$ else $Q \neq$ if $C$ then $(C;P)$ else $Q$
  (choose $C = \text{skip}$ or fail and $P, Q = \text{skip}$)
Part V

Case Study in Verification
Part VI

Miscellanea
Rooted programs

Example: rooted 2-colouring

Program implements a depth-first search.
Example: rooted 2-colouring

Main = try (init; Colour!; if grey_root then fail)
Colour = (ColourNode; try Invalid then break)!; try Back else break
ColourNode = \{colour_blue, colour_red\} Invalid = \{joined_blues, joined_reds\}
Back = \{back_blue, back_red\}

init(x:list)
\[ \begin{array}{c}
  1 \\
  x \\
  \Rightarrow \\
  1 \\
  x
\end{array} \Rightarrow \begin{array}{c}
  1 \\
  x \\
  \Rightarrow \\
  1 \\
  x
\end{array} \]

colour_blue(a,x,y:list)
\[ \begin{array}{c}
  1 \\
  x \\
  \Rightarrow \\
  1 \\
  y
\end{array} \Rightarrow \begin{array}{c}
  1 \\
  x \\
  \Rightarrow \\
  1 \\
  y
\end{array} \]

joined_reds(a,x,y:list)
\[ \begin{array}{c}
  1 \\
  x \\
  \Rightarrow \\
  2 \\
  y
\end{array} \Rightarrow \begin{array}{c}
  1 \\
  x \\
  \Rightarrow \\
  2 \\
  y
\end{array} \]

grey_root(x:list)
\[ \begin{array}{c}
  1 \\
  x \\
  \Rightarrow \\
  1 \\
  x
\end{array} \]

colour_red(a,x,y:list)
\[ \begin{array}{c}
  1 \\
  x \\
  \Rightarrow \\
  2 \\
  y
\end{array} \Rightarrow \begin{array}{c}
  1 \\
  x \\
  \Rightarrow \\
  2 \\
  y
\end{array} \]

back_red(a,x,y:list)
\[ \begin{array}{c}
  1 \\
  x \\
  \Rightarrow \\
  2 \\
  y
\end{array} \Rightarrow \begin{array}{c}
  1 \\
  x \\
  \Rightarrow \\
  2 \\
  y
\end{array} \]
Runtime of rooted 2-colouring: GP 2 vs C

Comparison with Sedgewick’s tailor-made program from *Algorithms in C* (Addison-Wesley, 2002)

3 × 3 square grid

Star graph of degree 4
Runtime of rooted 2-colouring: GP 2 vs C

Square Grids

Star Graphs
In general, the GP 2 program runs in linear time on bounded-degree graphs.
Other topics in graph programs

- Probabilistic graph programs for randomised and evolutionary algorithms [Atkinson-P-Stepney 18a,18b]
- Checking confluence by critical-pair analysis [Hristakiev-P 15,17,18; Hristakiev 17]
- Computational completeness [P 17]
- Compiling GP 2 to C [Bak 15; Bak-P 16]
- Hoare-style program verification [Poskitt-P 10,12a,12b,14; Poskitt 13; P 16; Wulandari-P 18]
- Case study in automata minimization [P-Suri-Singh 11]
Part VII

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- Chris Poskitt: *Verification of Graph Programs*, 2013
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