Critical Pairs in Graph Transformation Systems

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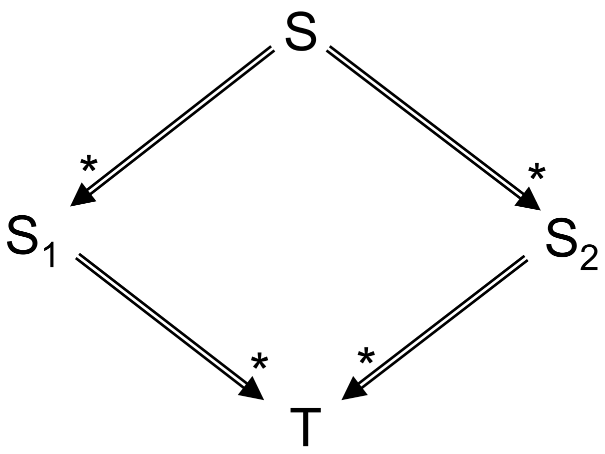
Work in cooperation, especially with Leen Lambers (Hasso Plattner Institut) and Hartmut Ehrig (passed away in 2016)

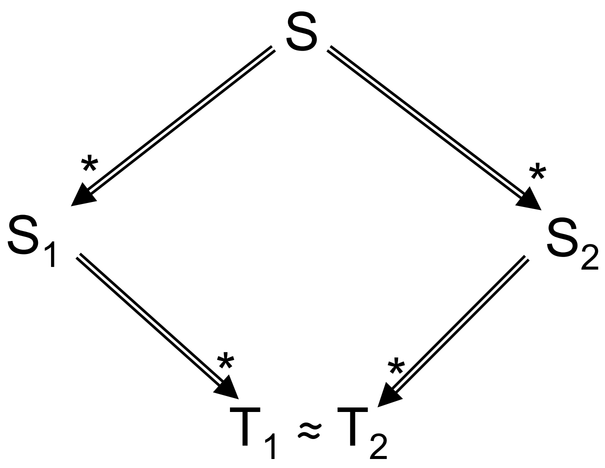
1. Introduction
2. Preliminaries on Graph Transformation
3. Critical Pairs for GTS
4. Critical Pairs for conditional rules with NACs
5. Essential Critical Pairs
6. Initial Critical Pairs

Introduction

Confluence and Local Confluence

(Local) Confluence is a property of rule-based systems needed especially when they are used in deterministic computations





Critical Pairs and Local Confluence

Critical pairs are minimal representations of two conflicting (possibly non-confluent) transformations.

Confluence and Local Confluence

Confluence and local confluence are undecidable for graph transformation systems, even if they are terminating.

D. Plump (1993)

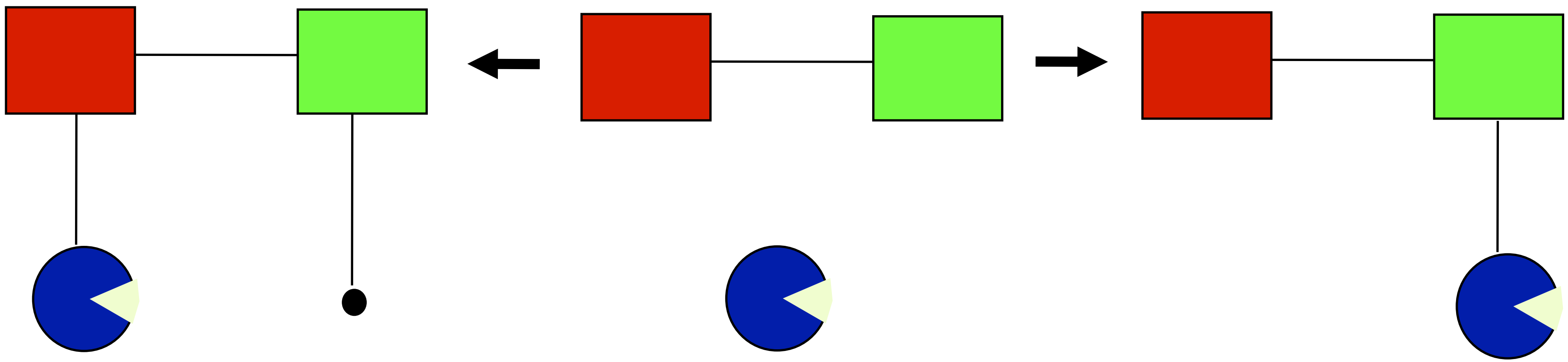
Preliminaries on Graph Transformation

Double Pushout Graph Transformation

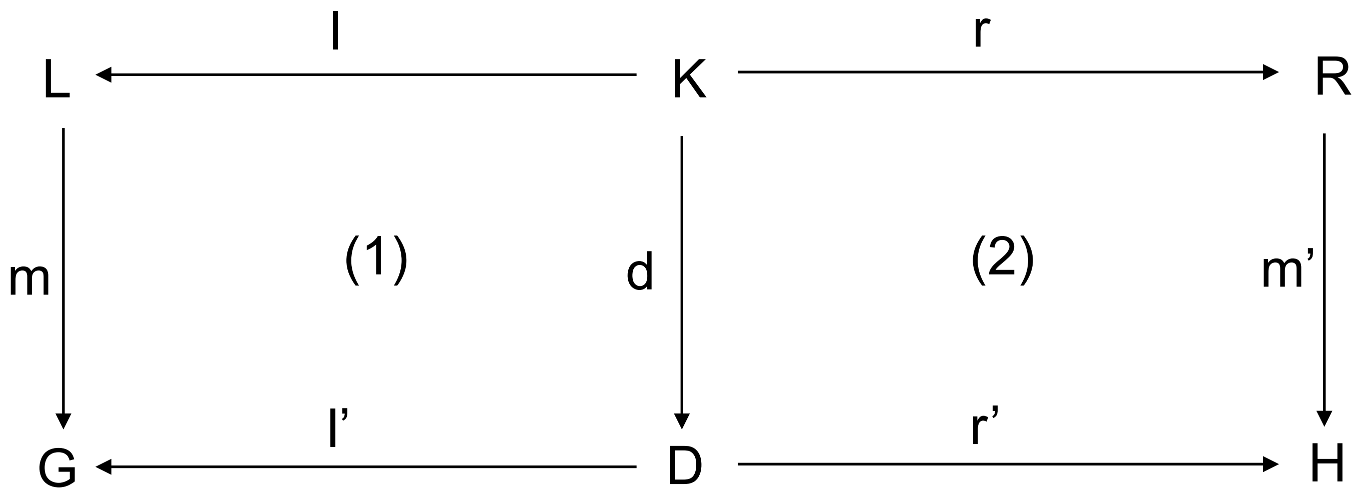
Rules:

$$p = L \xleftarrow{l} K \xrightarrow{r} R$$

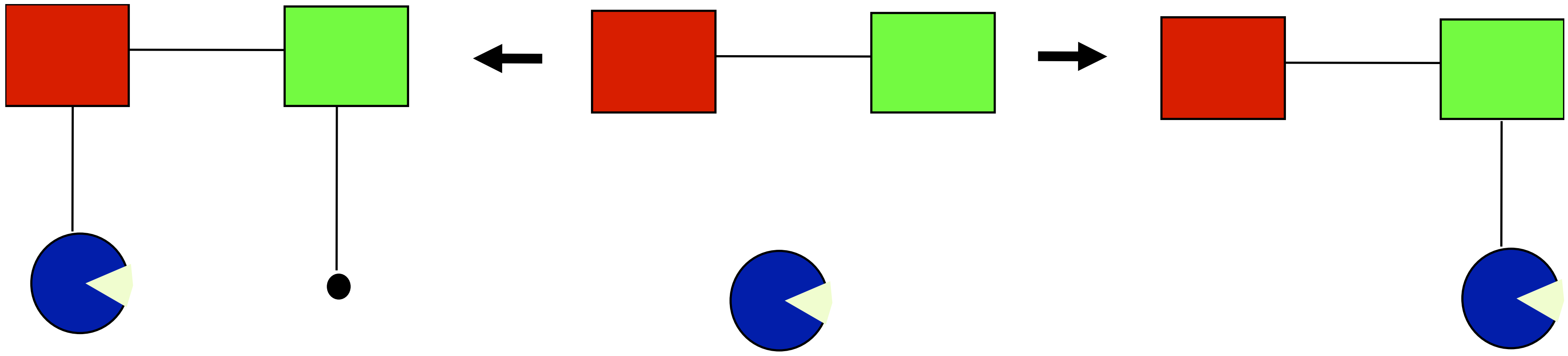
A graph transformation rule:

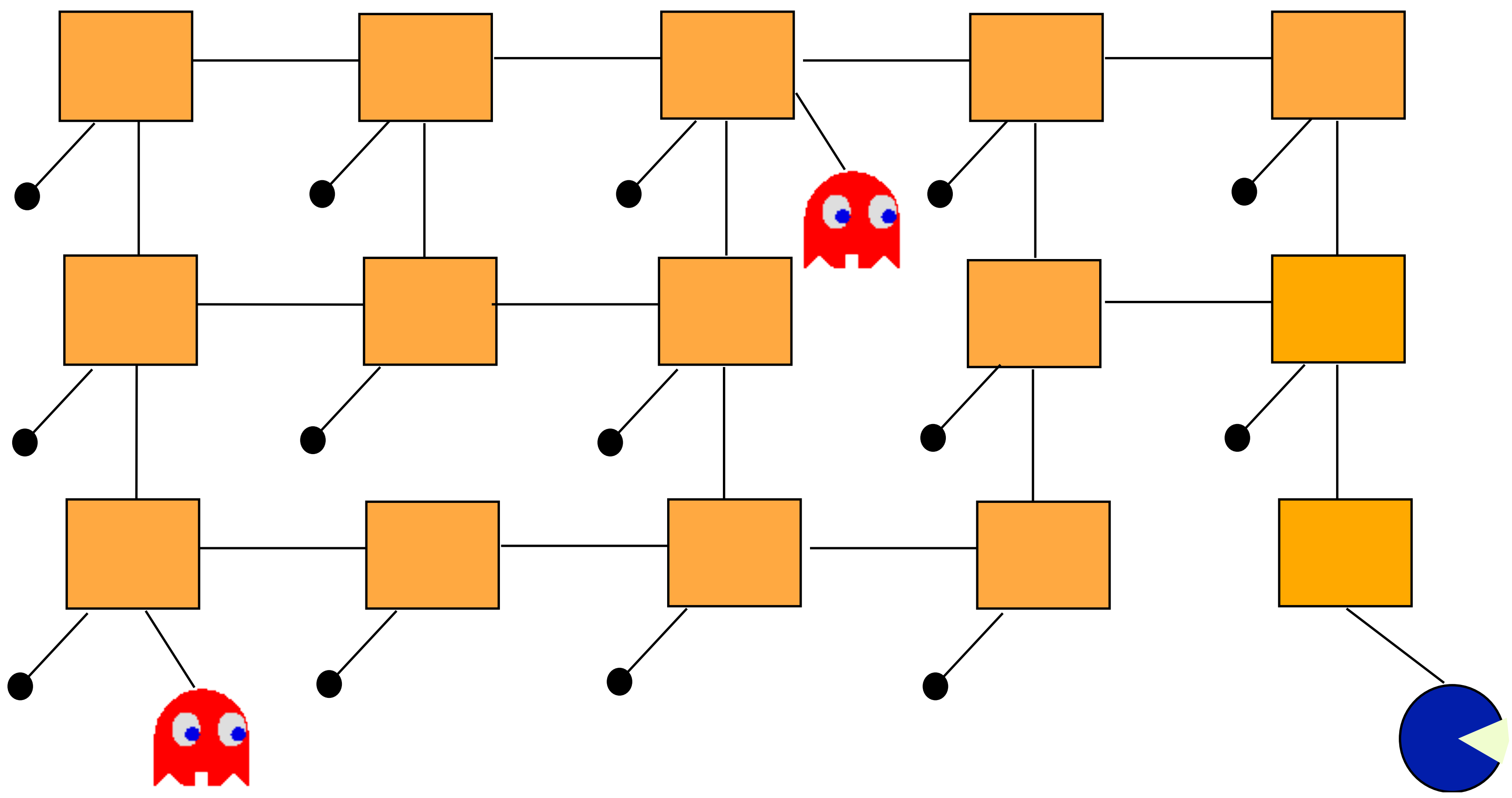


Rule Application:



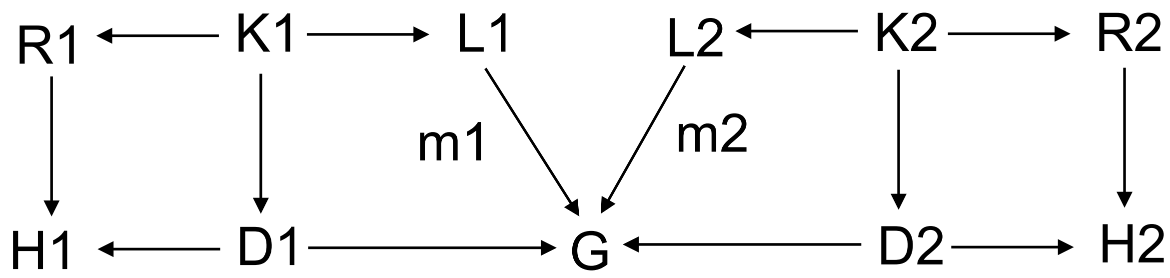
Such that (1) and (2) are pushouts





Parallel Independence

Two rule applications on the same graph G:

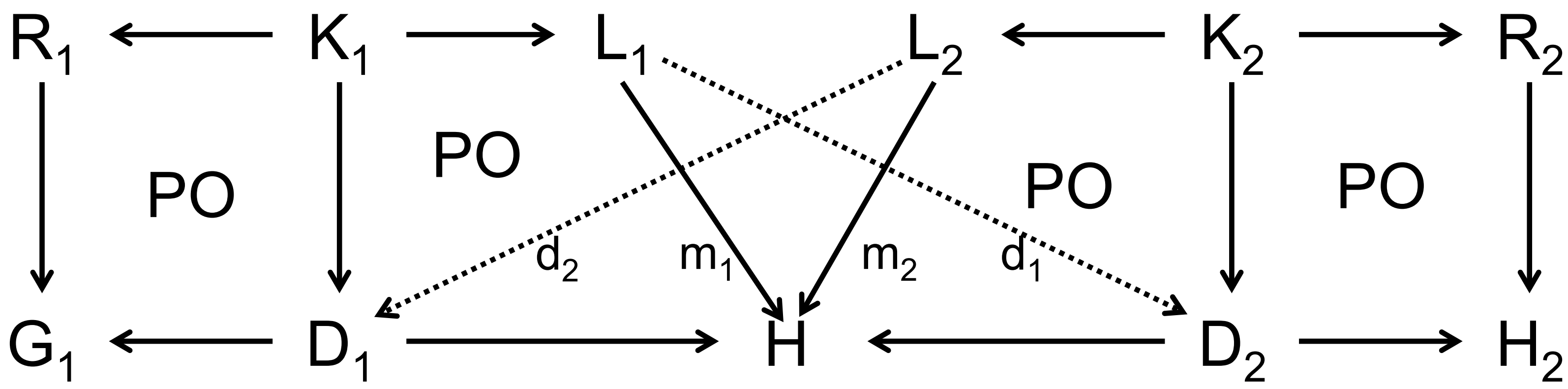


are parallel independent if

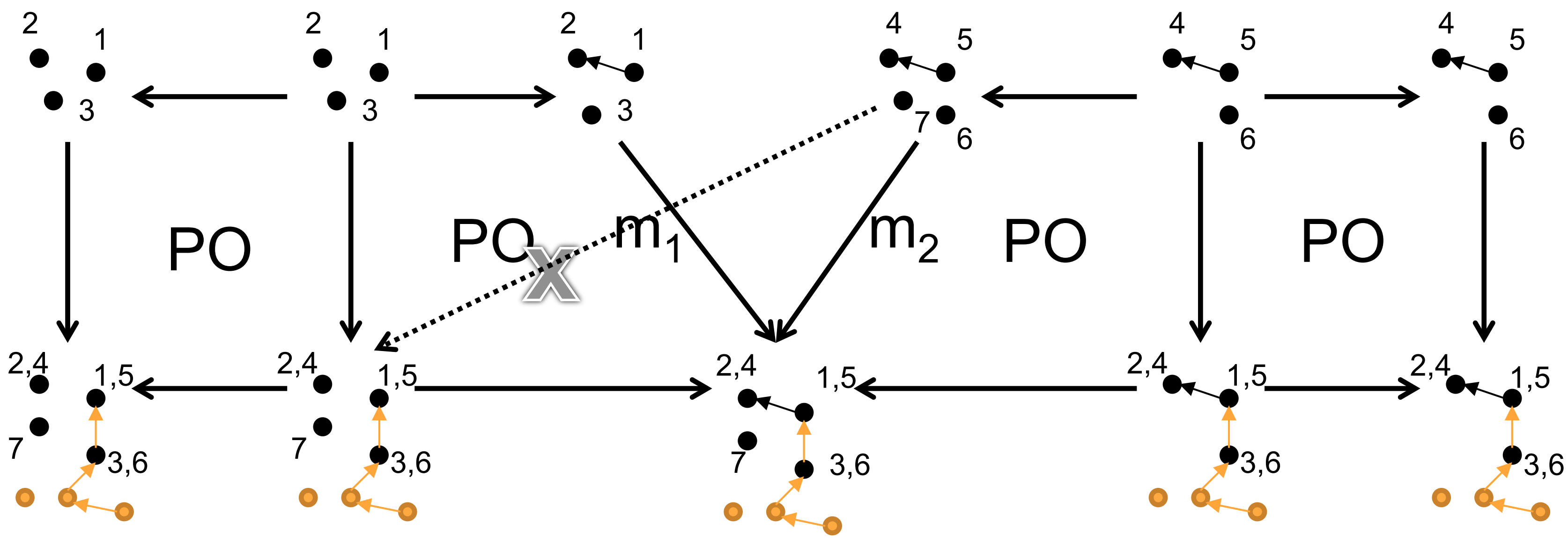
$$(m1(L1) \cap m2(L2)) \subseteq (m1(K1) \cap m2(K2))$$

Parallel Independence

or equivalently if

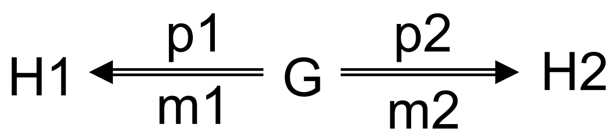


Example of non-parallel independence

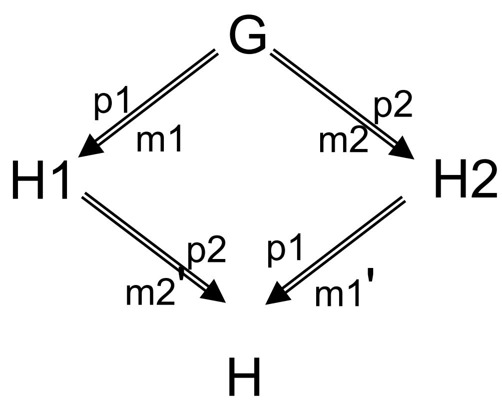


Local Church-Rosser Theorem

Given two parallel independient applications of rules p1 and p2 on a graph G:



there are applications of p1 to H2 and of p2 to H1 such that the final result H coincides:

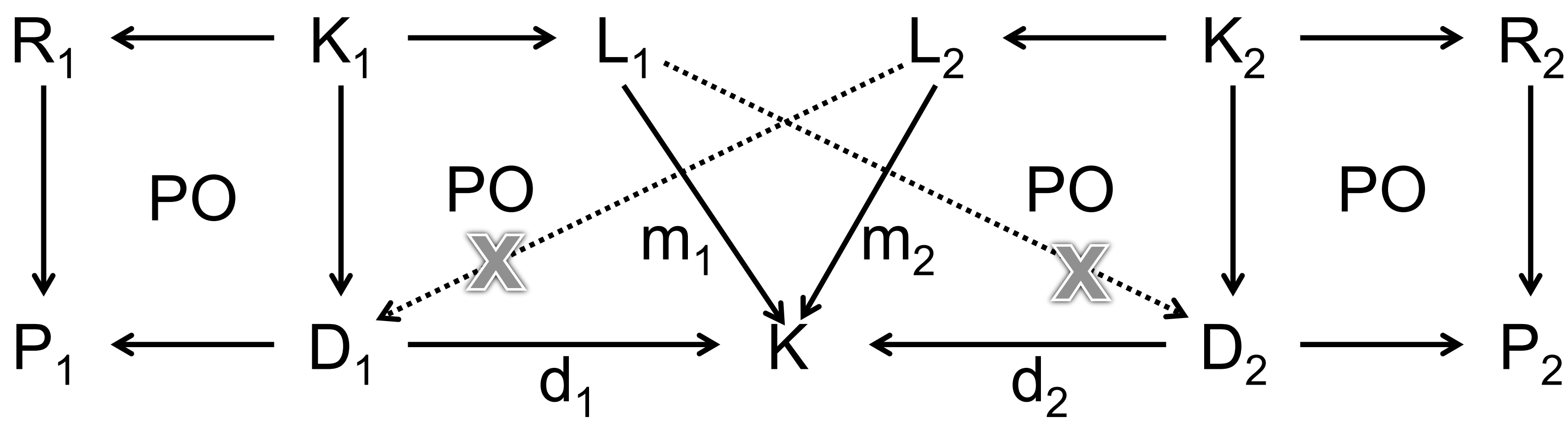


where m1' and m2' are, respectively, extensions of m1 and m2.

Critical Pairs for GTS

Critical Pairs

A critical pair is a minimal conflict, i.e. two direct transformations



such that m_1: $L_1 \rightarrow K$ and m_2: $L_2 \rightarrow K$ are jointly surjective and there is no morphism h_1: $L_1 \rightarrow D_2$ with $d_2 \cdot h_1 = m_1$, or there is no morphism h_2: $L_2 \rightarrow D_1$, with $d_1 \cdot h_2 = m_2$.

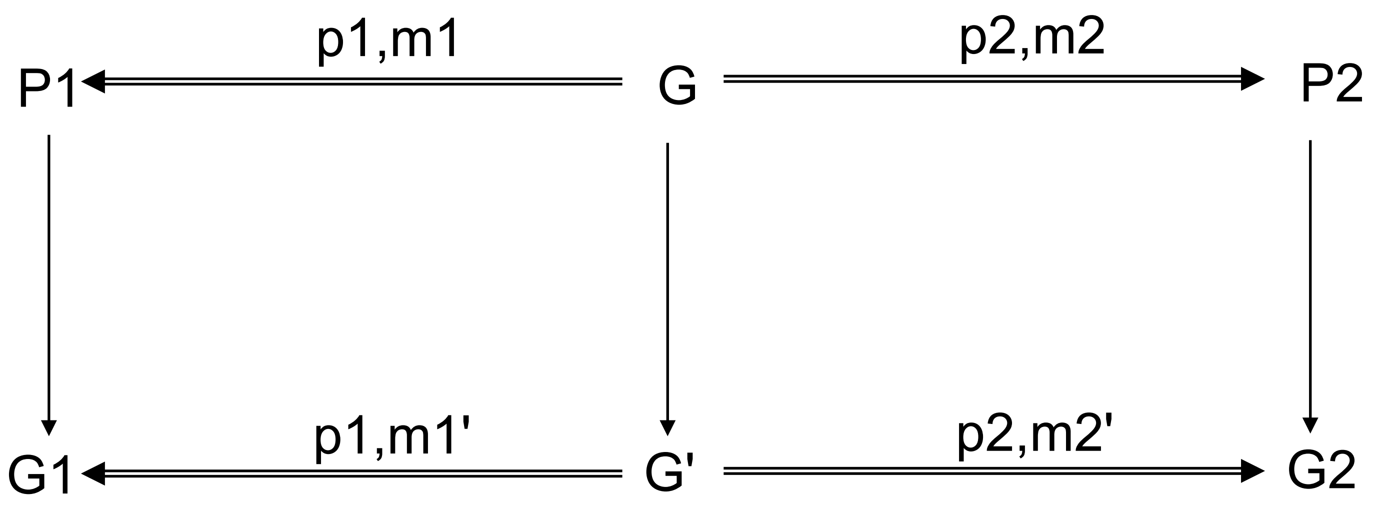
D. Plump (1993)

Completeness Theorem

For any two parallel dependent direct transformations:

$$G1 \; {}_{p1,m1'}\!\Leftarrow G' \Rightarrow_{p2,m2'} G2$$

there is a critical pair $P1 \; {}_{p1,m1}\!\Leftarrow G \Rightarrow_{p2,m2} P2$ such that:



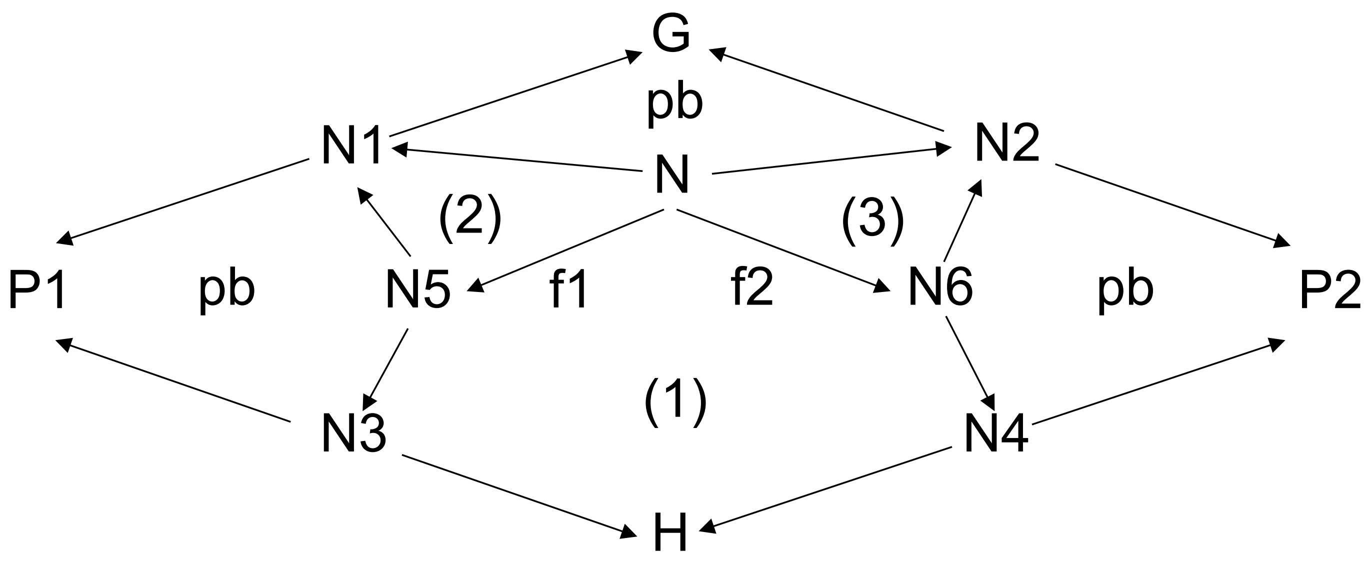
Critical Pairs

Two main problems:

- Confluence of critical pairs does not guarantee the local confluence of the system, it is only a necessary condition.
- In general, there is an exponential number of critical pairs

Strict Confluence

$P1 \;_{p1,m1}\!\Leftarrow G \Rightarrow_{p2,m2} P2$ is strictly confluent if there are morphisms f1 and f2 such that (1), (2), and (3) commute.

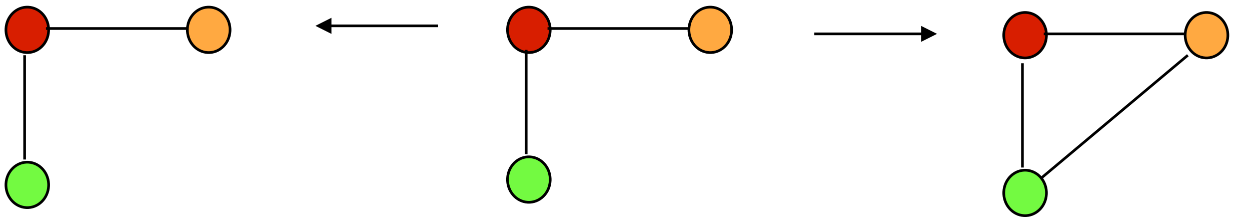


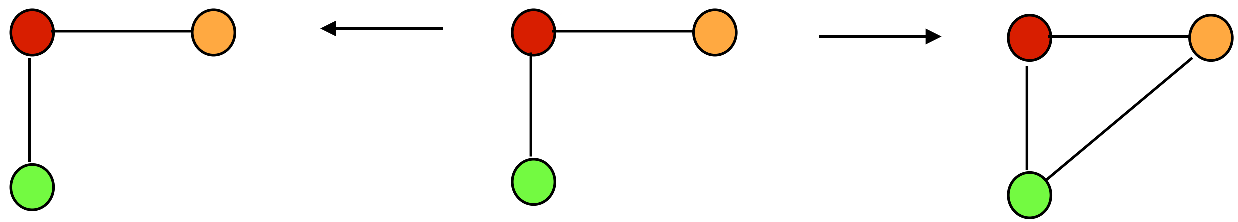
Local Confluence Theorem

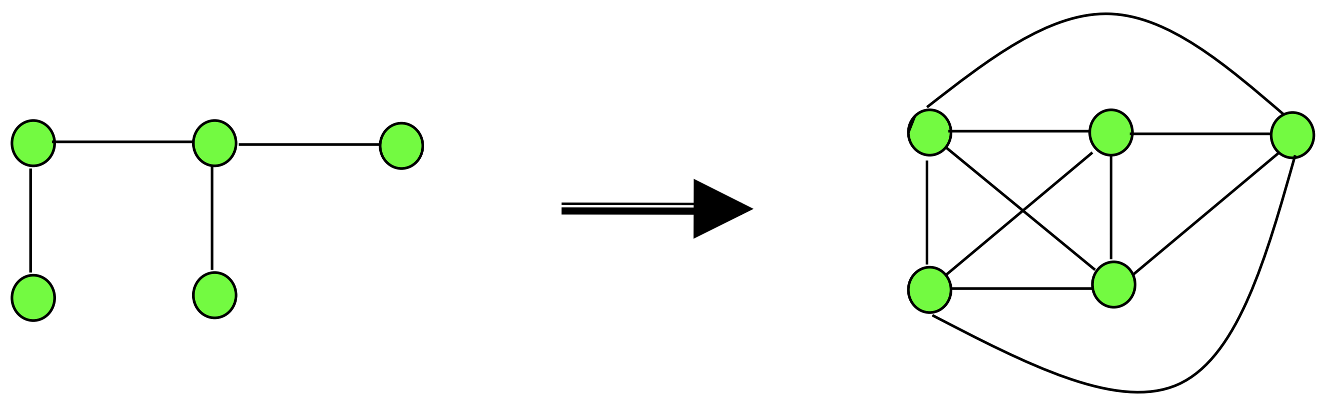
A graph transformation system is locally confluent if all its critical pairs are strictly confluent.

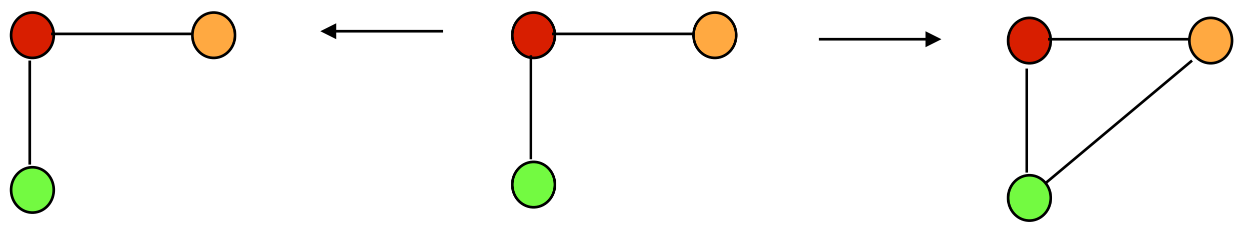
Critical Pairs for Conditional Rules with NACs

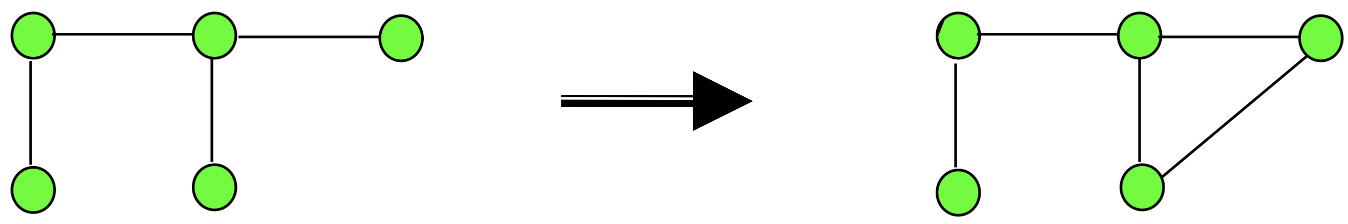
Negative Application Conditions (NACs)

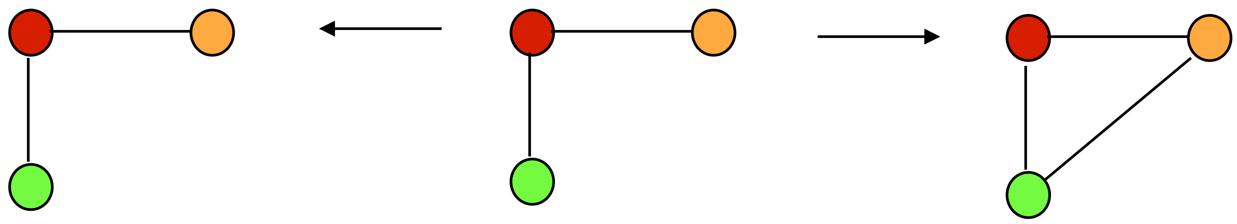


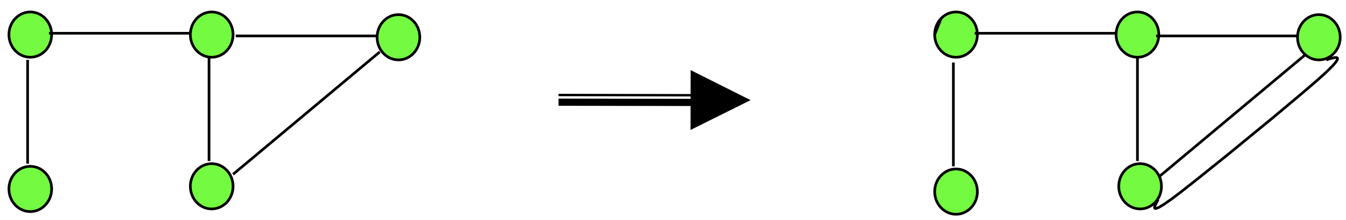


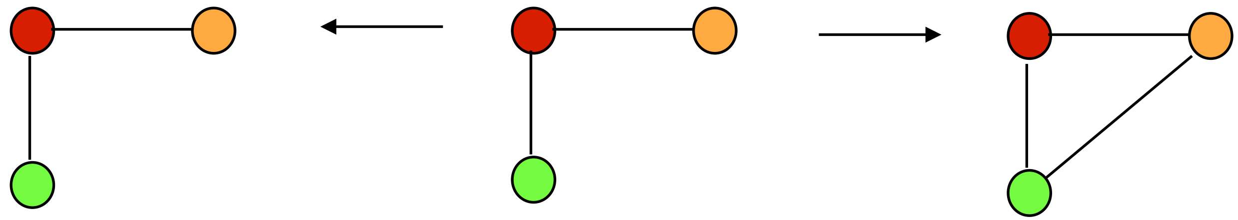


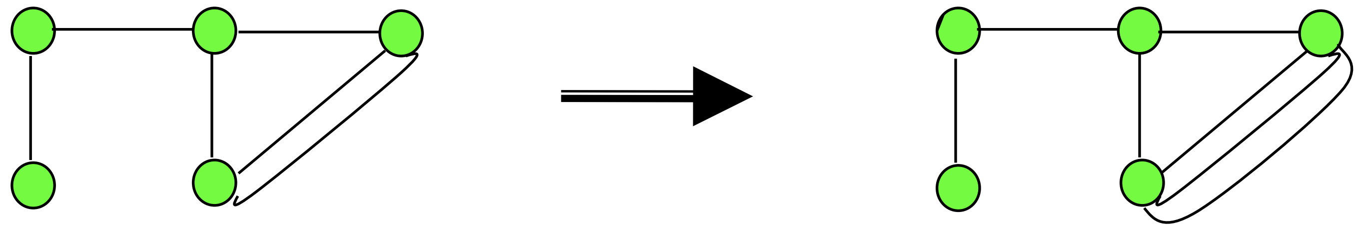


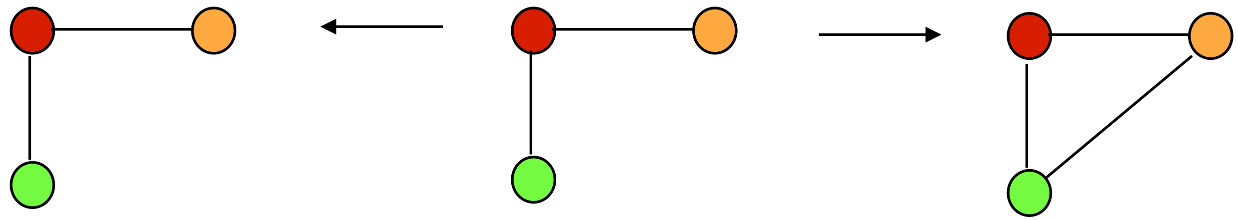


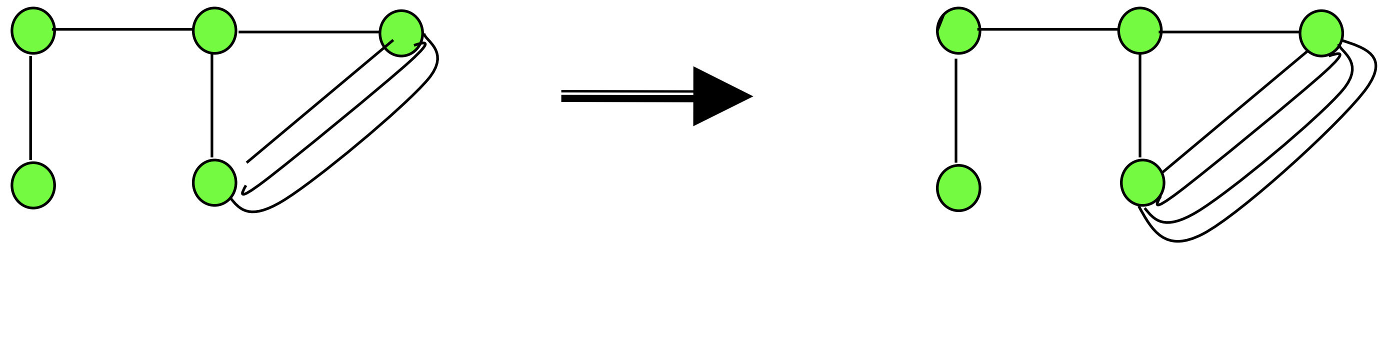












Negative Application Conditions (NACs)

Given a rule

$$p = L \xleftarrow{l} K \xrightarrow{r} R$$

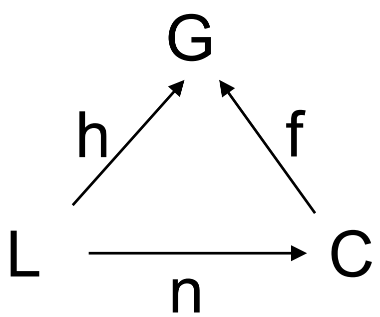
a negative application condition is an inclusion

$$L \rightarrow C$$

NAC satisfaction:

$(h: L \to G) \models L \to C$

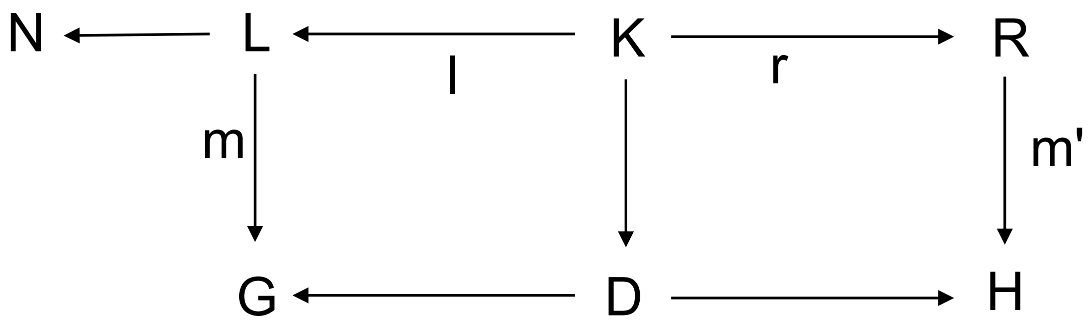
there is no $f: C \to G$ such that the diagram commutes



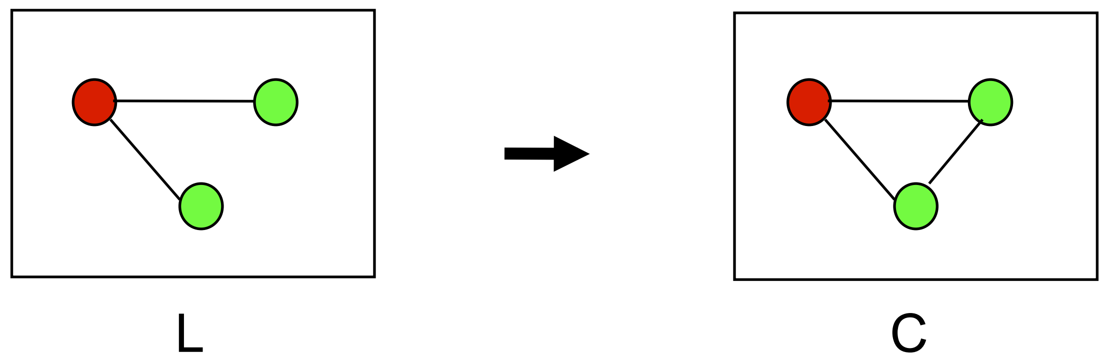
Graph Transformation with NACs

Given the rule $p = L \xleftarrow{l} K \xrightarrow{r} R$

together with a set of NACs N, we can apply p to G vía m:



if m satisfies all the NACs in N


L
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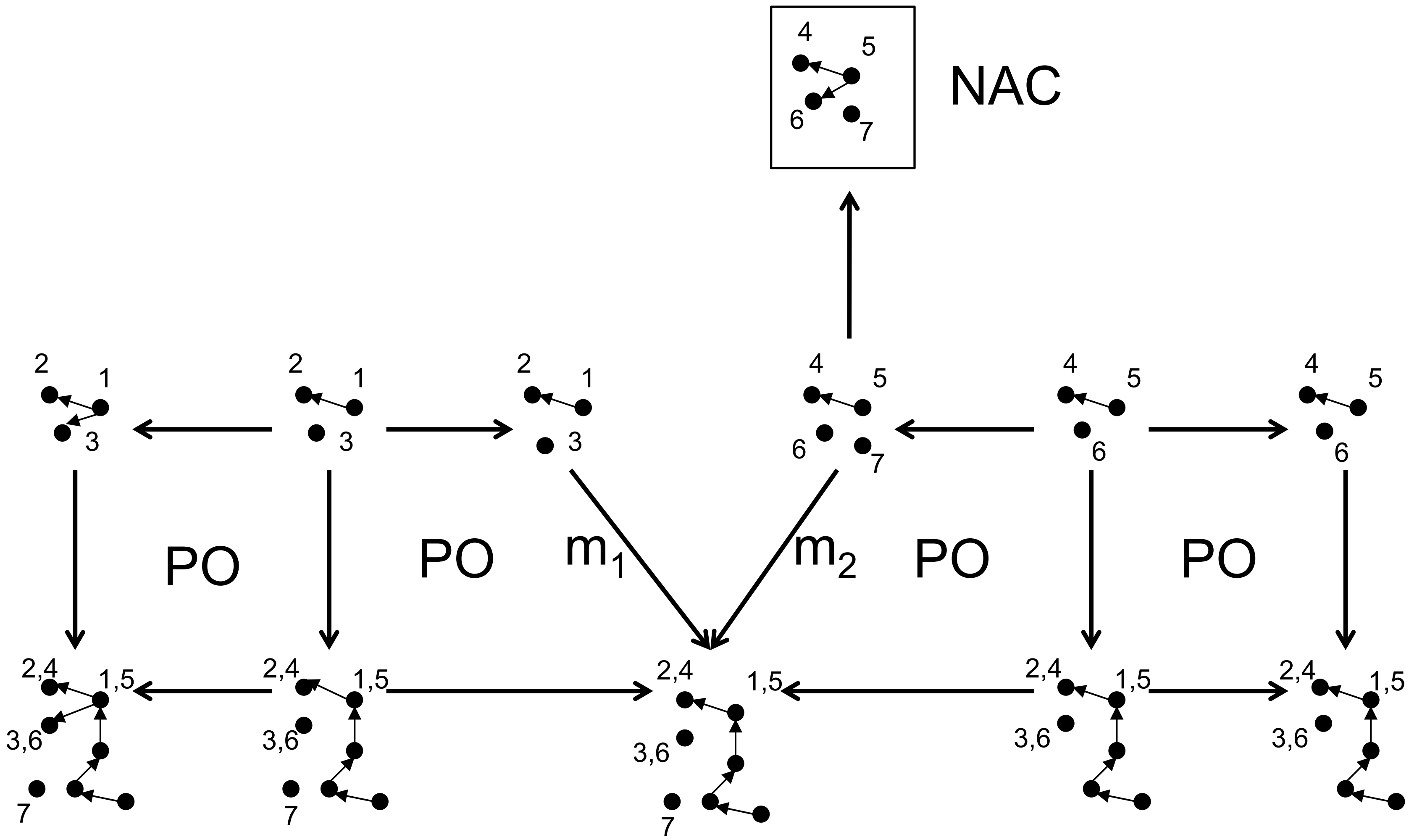
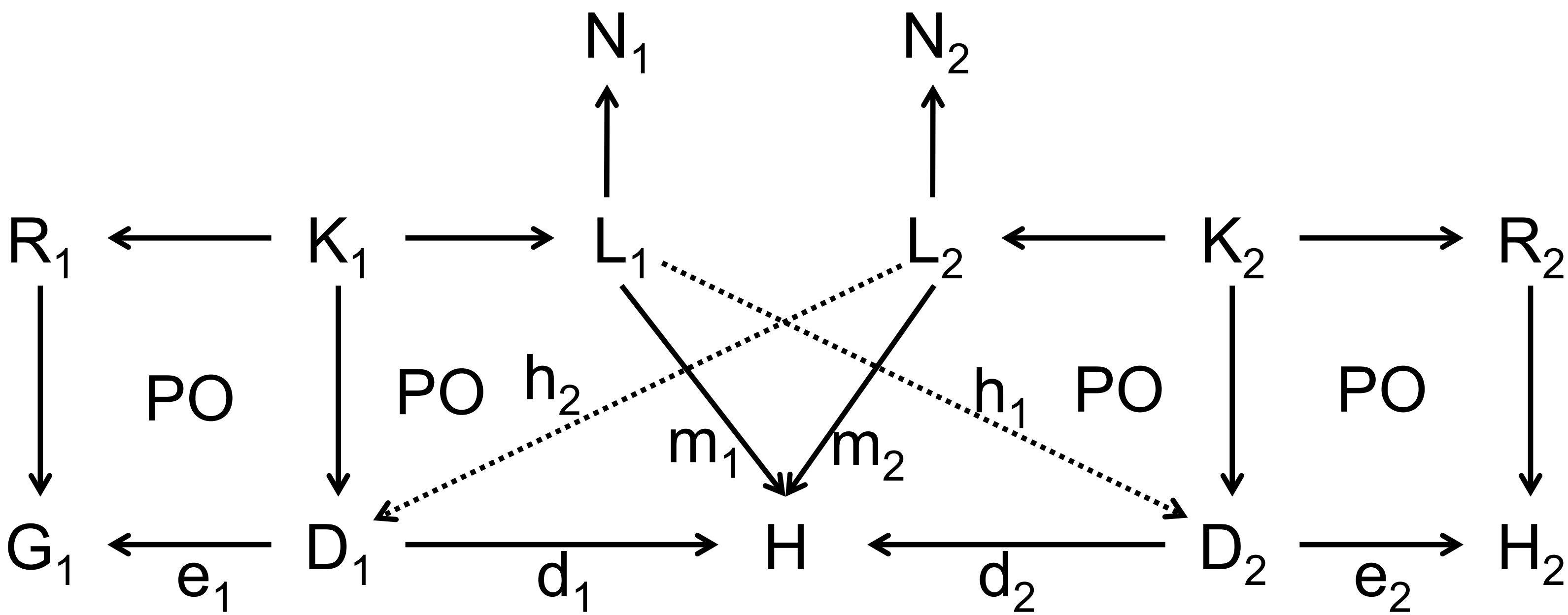
Parallel Independence



Parallel Independence with NACs

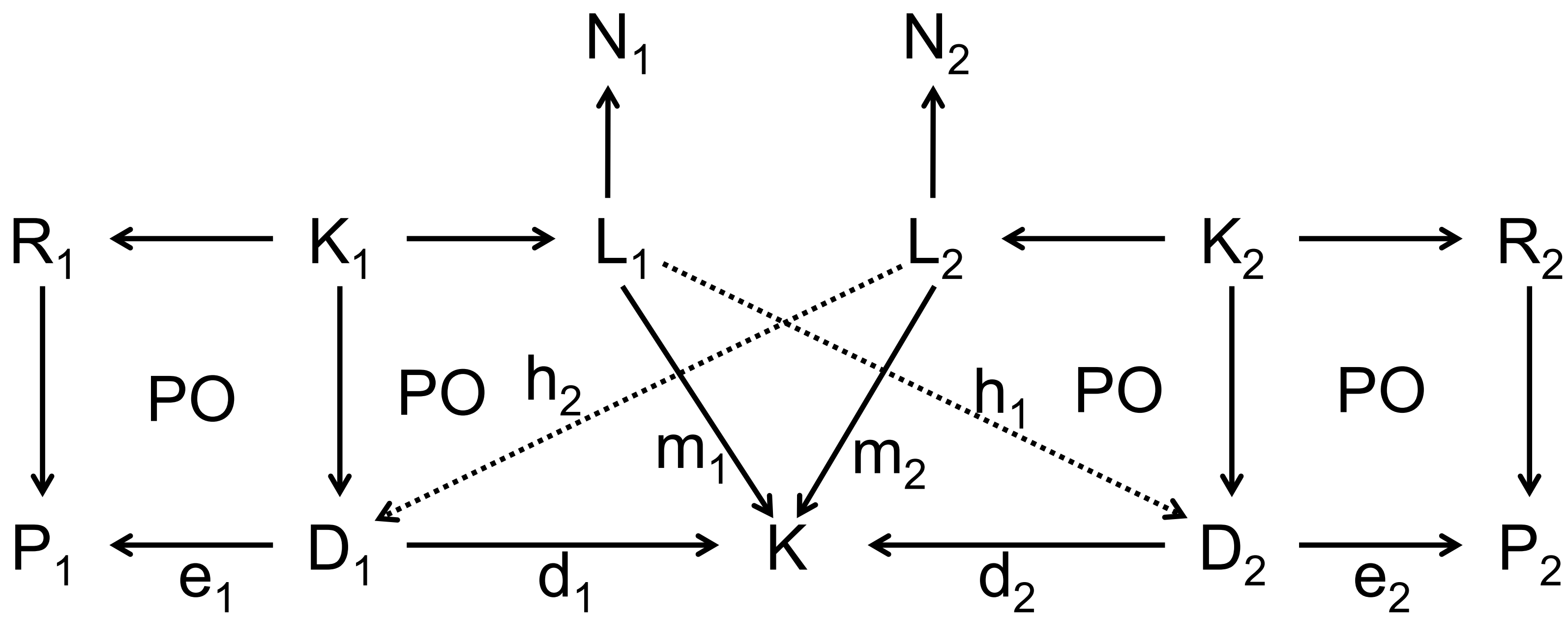
Two transformations with NACs on the same graph G are parallel independent:



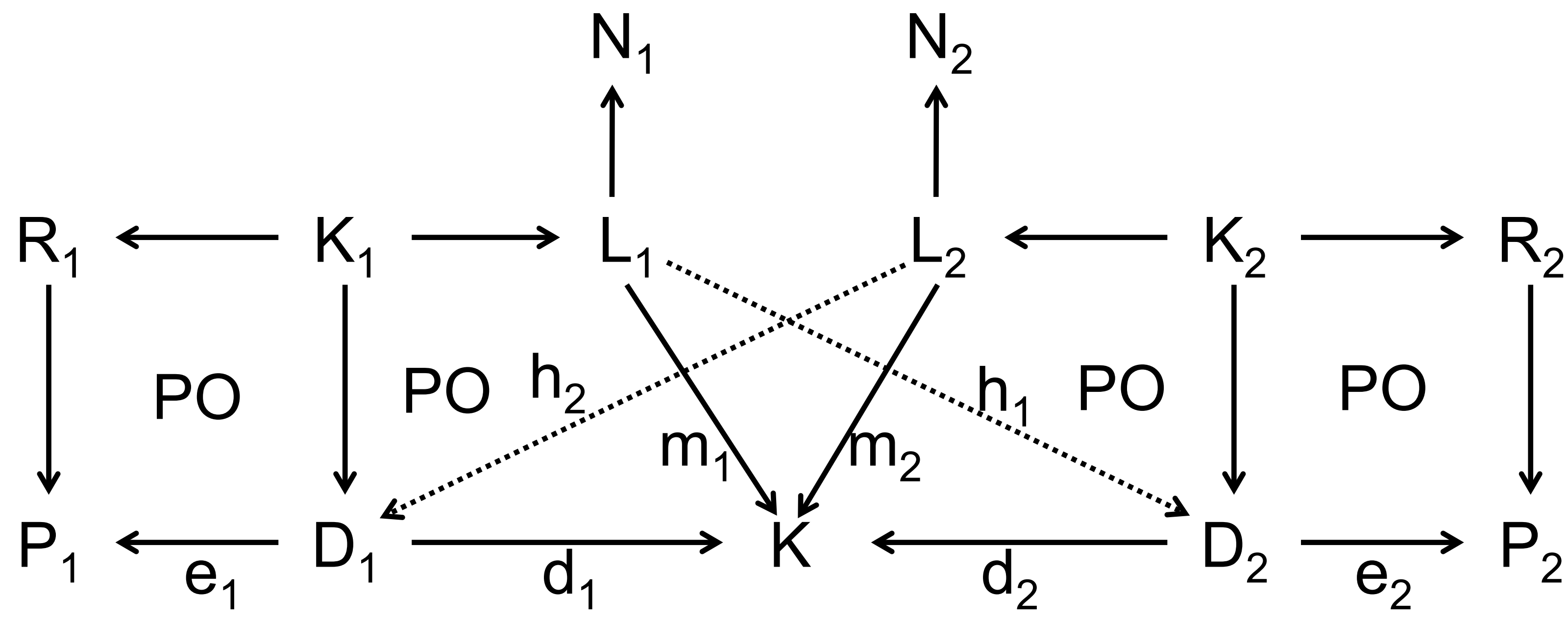
if there exist morphisms h_1 and h_2 such that $d_2 \cdot h_1 = m_1$, $d_1 \cdot h_2 = m_2$, $e_1 \cdot h_2 \models N_2$, and $e_2 \cdot h_1 \models N_1$

Critical Pairs for Rules with NACs

A critical pair consists of two transformations with NACs on the same graph K such that m_1 and m_2 are jointly surjective and one of the following conditions hold:



1. There is no morphism h_1 such that $d_2 \cdot h_1 = m_1$



2. There is no morphism h_2 such that $d_1 \cdot h_2 = m_2$
3. There is a morphism h_1 but $e_2 \cdot h_1 \models N_1$
4. There is a morphism h_2 but $e_1 \cdot h_2 \models N_2$

L. Lambers, H. Ehrig, F. Orejas (2006),

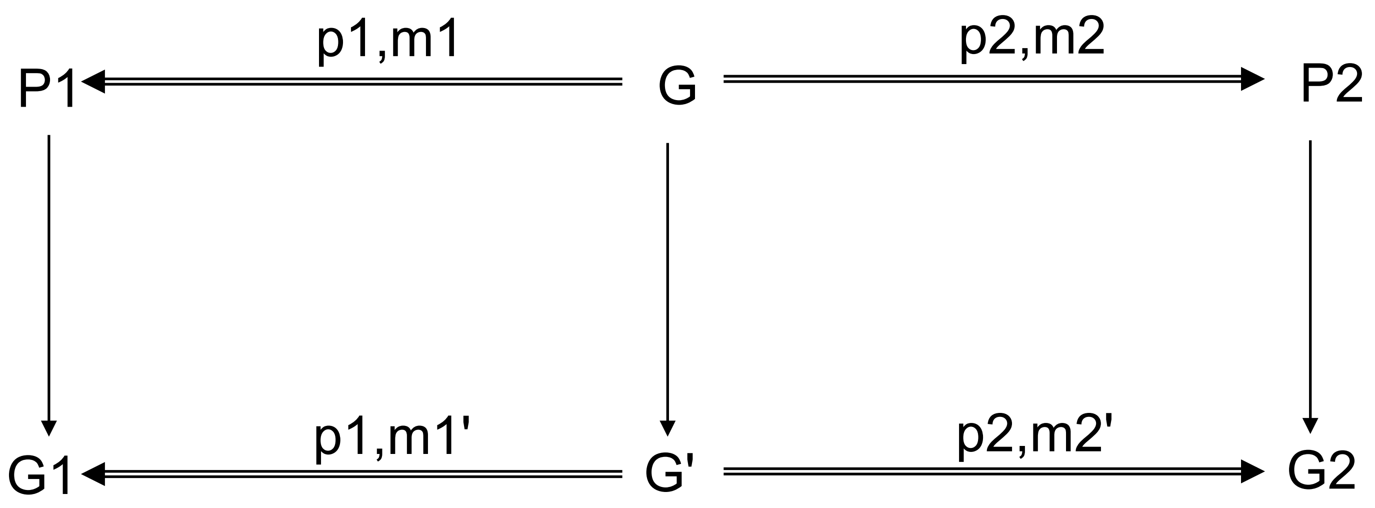
L. Lambers, H. Ehrig, U. Golas, F. Orejas (2008)

Completeness Theorem

For any two parallel dependent direct transformations with NACs:

$$G1 \; {}_{p1,m1'}\!\Leftarrow G' \Rightarrow_{p2,m2'} G2$$

there is a critical pair $P1 \; {}_{p1,m1}\!\Leftarrow G \Rightarrow_{p2,m2} P2$ such that:



Local Confluence Theorem

A graph transformation system with NACs is locally confluent if all its critical pairs are strictly NAC-confluent.

Elimination of conflicts

Given the critical pair $P1 \;_{p1,m1}\!\Leftarrow K \Rightarrow_{p2,m2} P2$, if we know that, in the situation where the conflict happens, the right rule to be applied is p1, then the conflict can be eliminated by adding to p2 the *filter NAC* $L2 \rightarrow K$.

F. Hermann, H. Ehrig, U. Golas, F. Orejas (2008)

Generalization

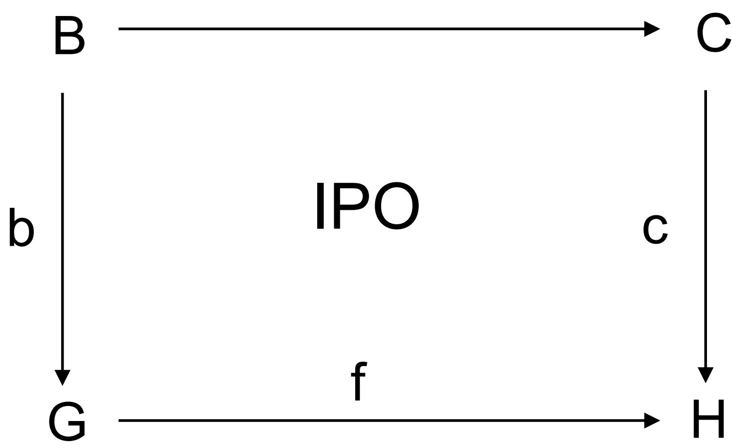
The previous results have been generalized to arbitrary M-adhesive categories where, in addition, NACs are arbitrary conditions written in the Logic of Nested Conditions (A. Habel and K.H Penneman (2009)), as expressive as First-Order Logic.

H. Ehrig, U. Golas, A. Habel, L. Lambers and F. Orejas (2012)

Essential Critical Pairs

Initial Pushouts

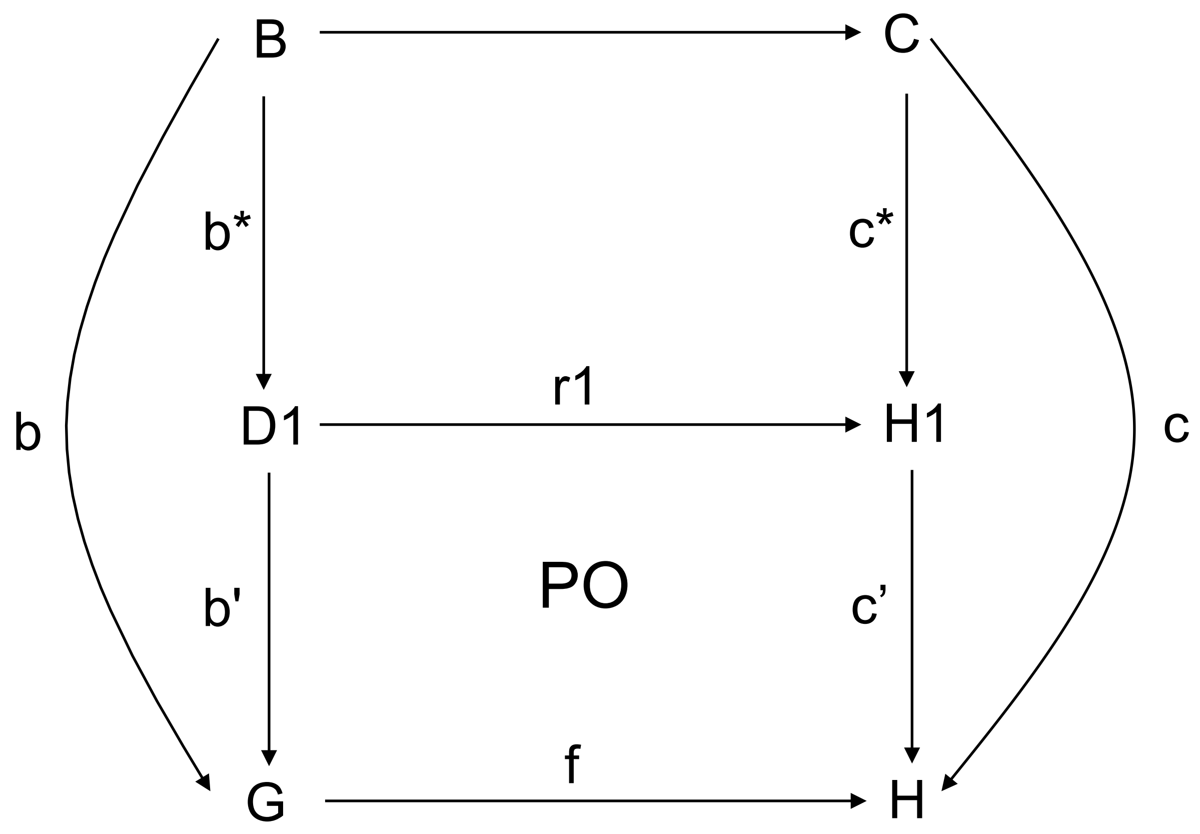
Given a morphism f: G → H, its initial pushout is defined as a pushout:



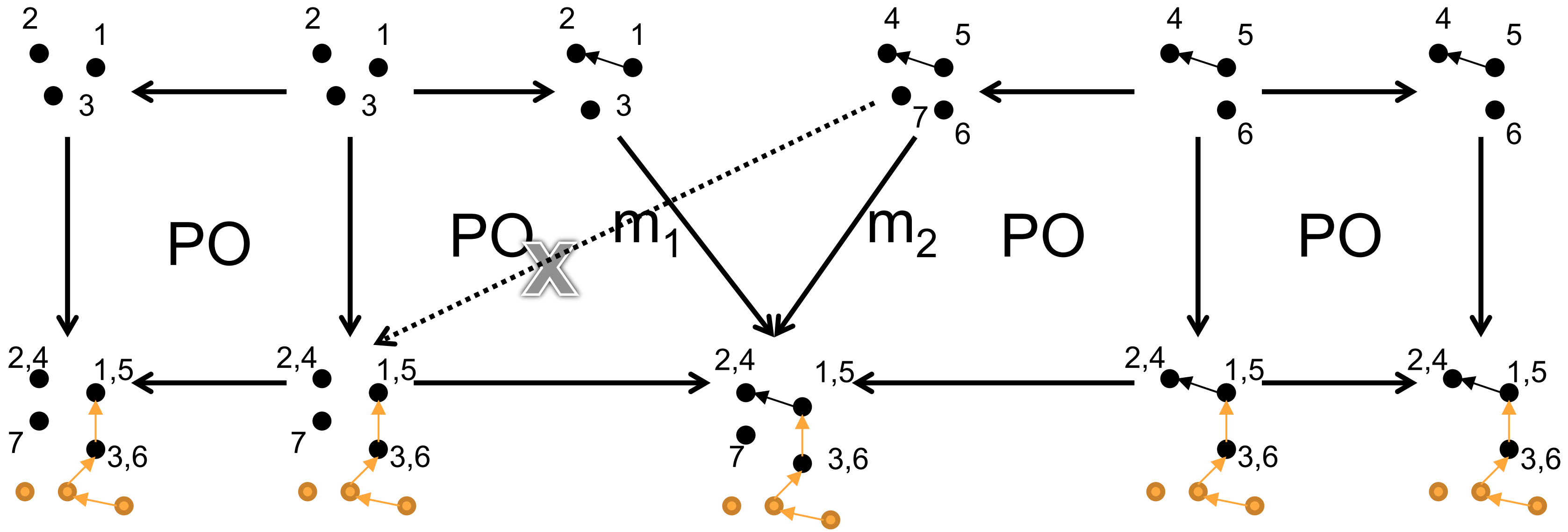
where C, called the *context*, is the smallest graph including the elements in H\f(G), and B, called the *boundary,* includes the "intersection" of C and G.

Initial Pushouts

Formally:

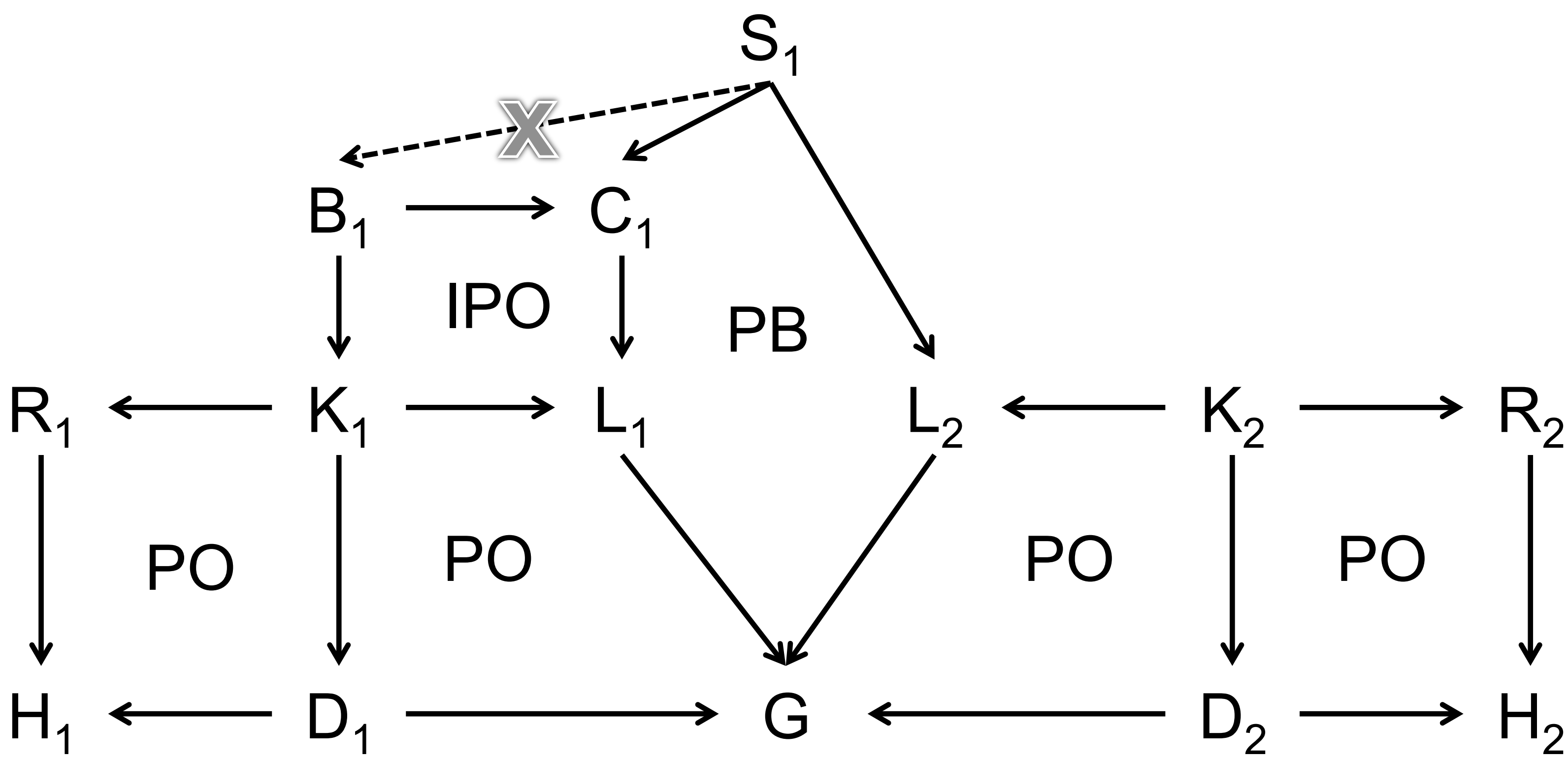


Example of a conflict



Conflict Reasons

Two transformations have a delete-use conflict if



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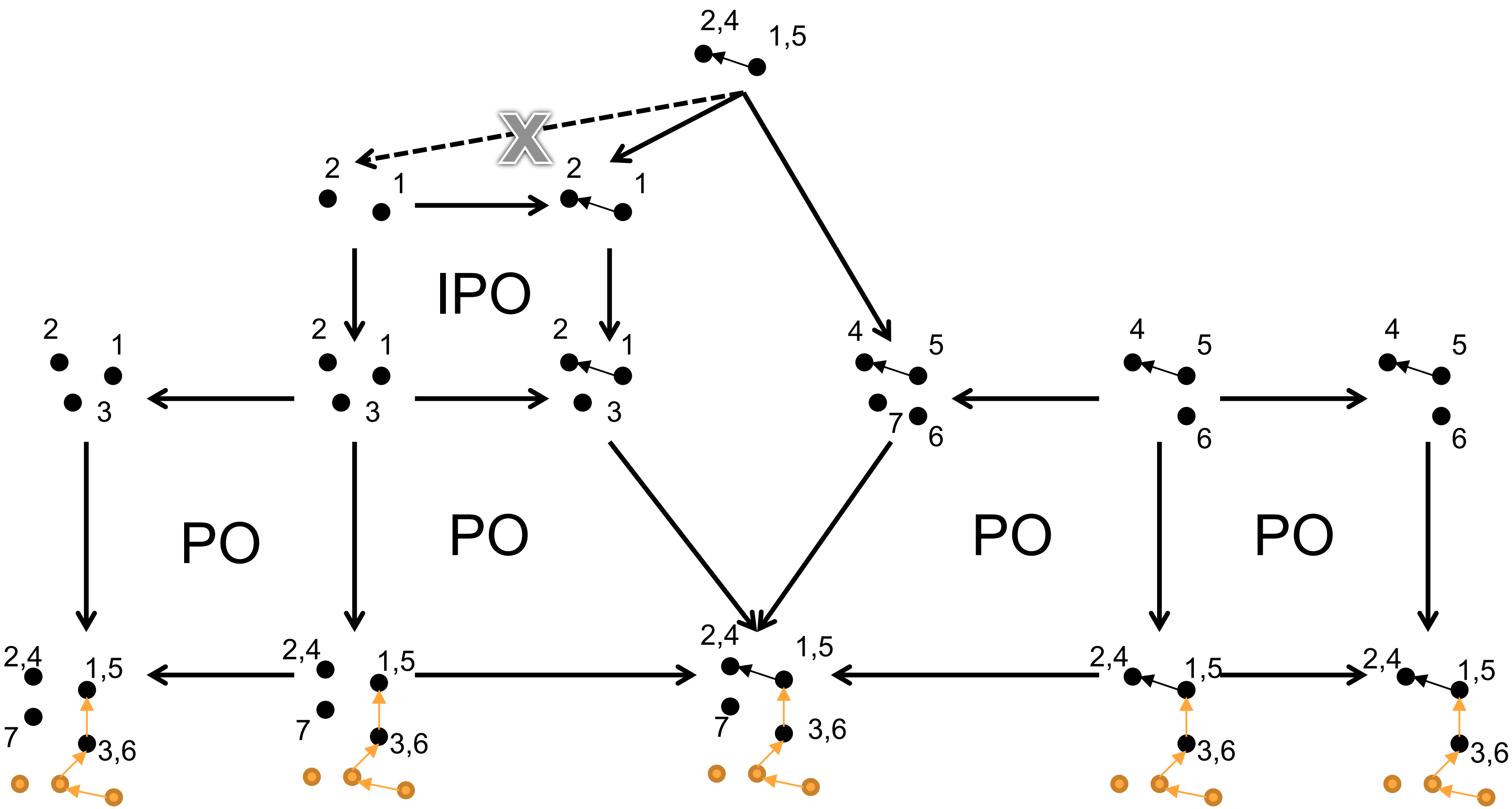

Conflict Reasons

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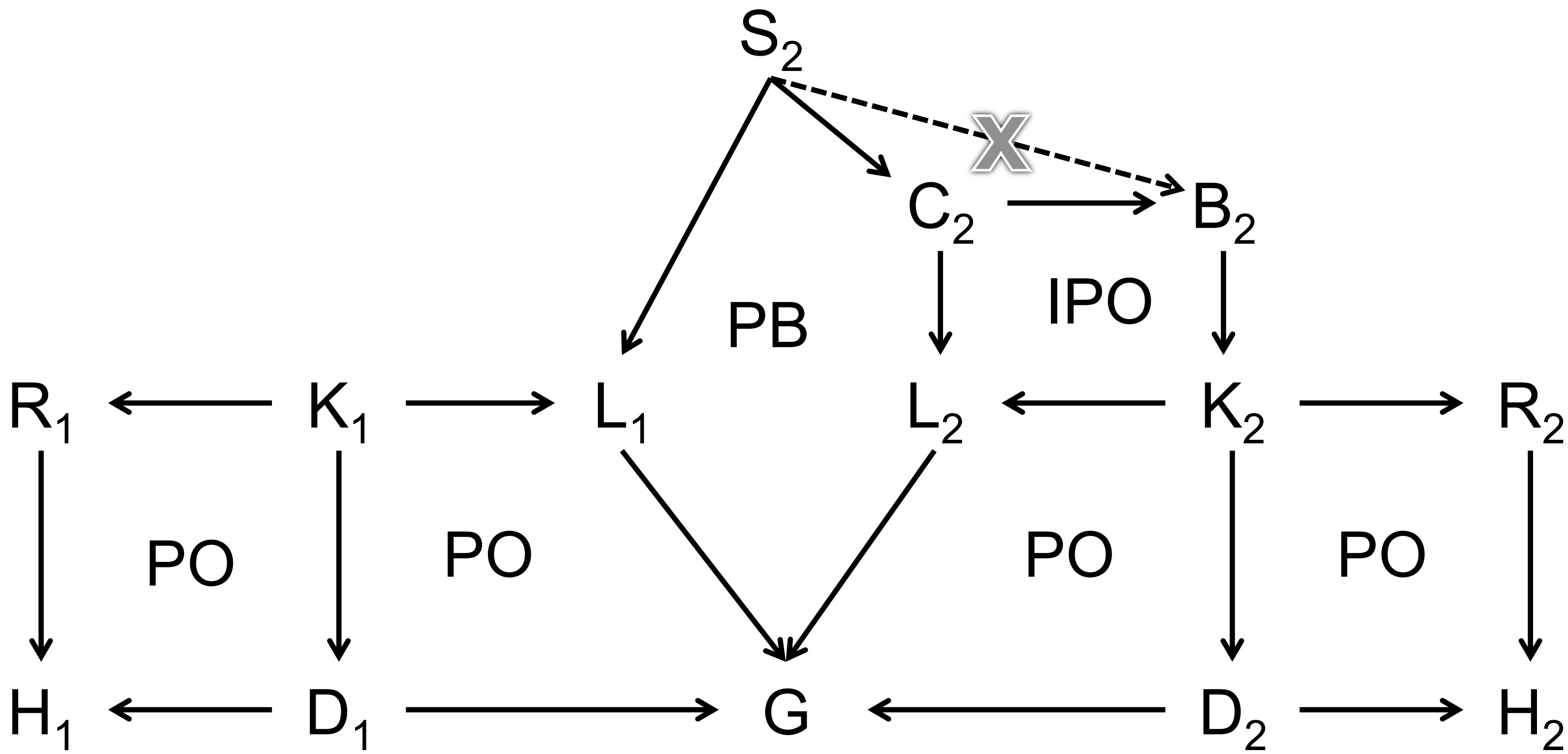
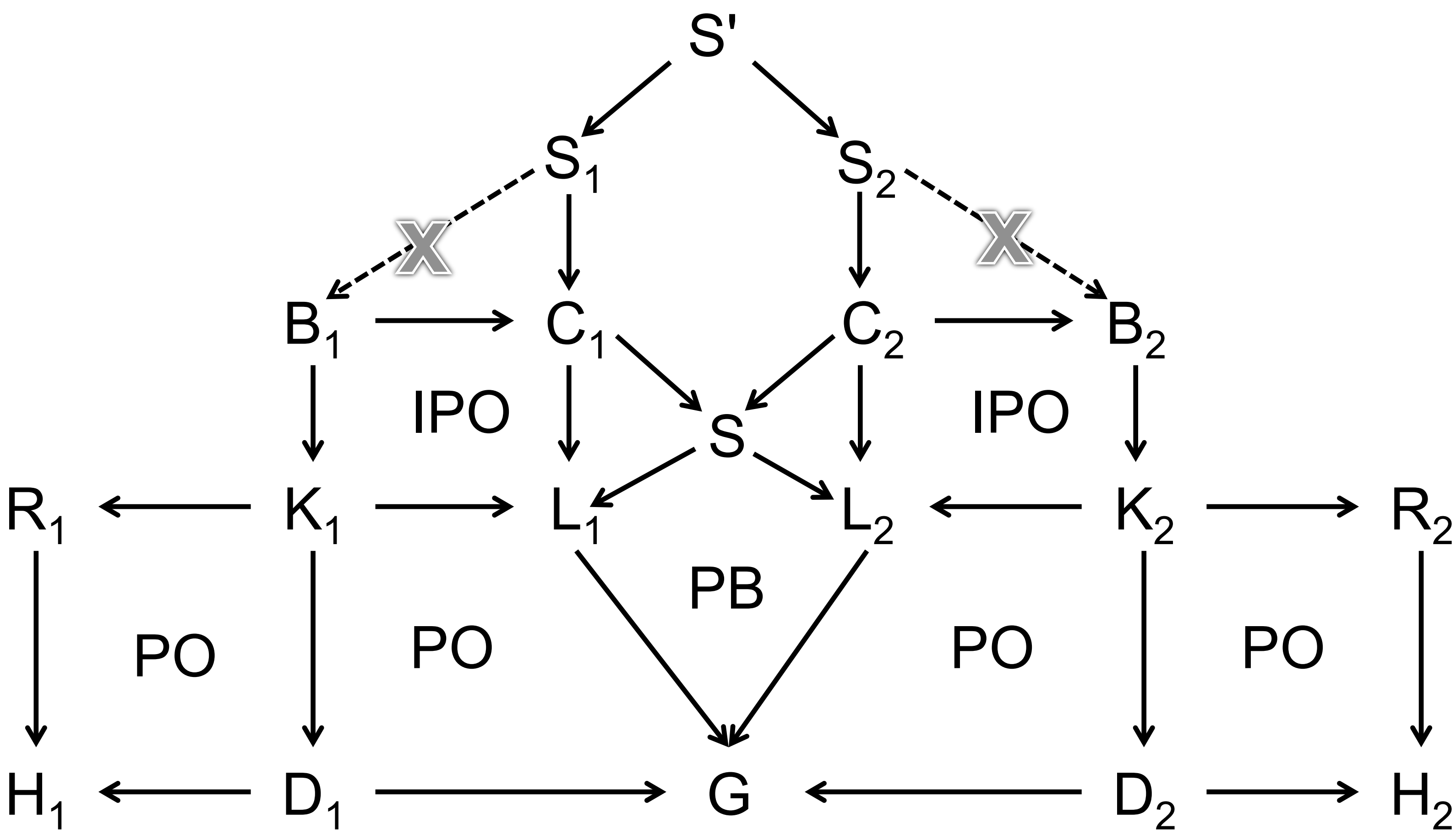


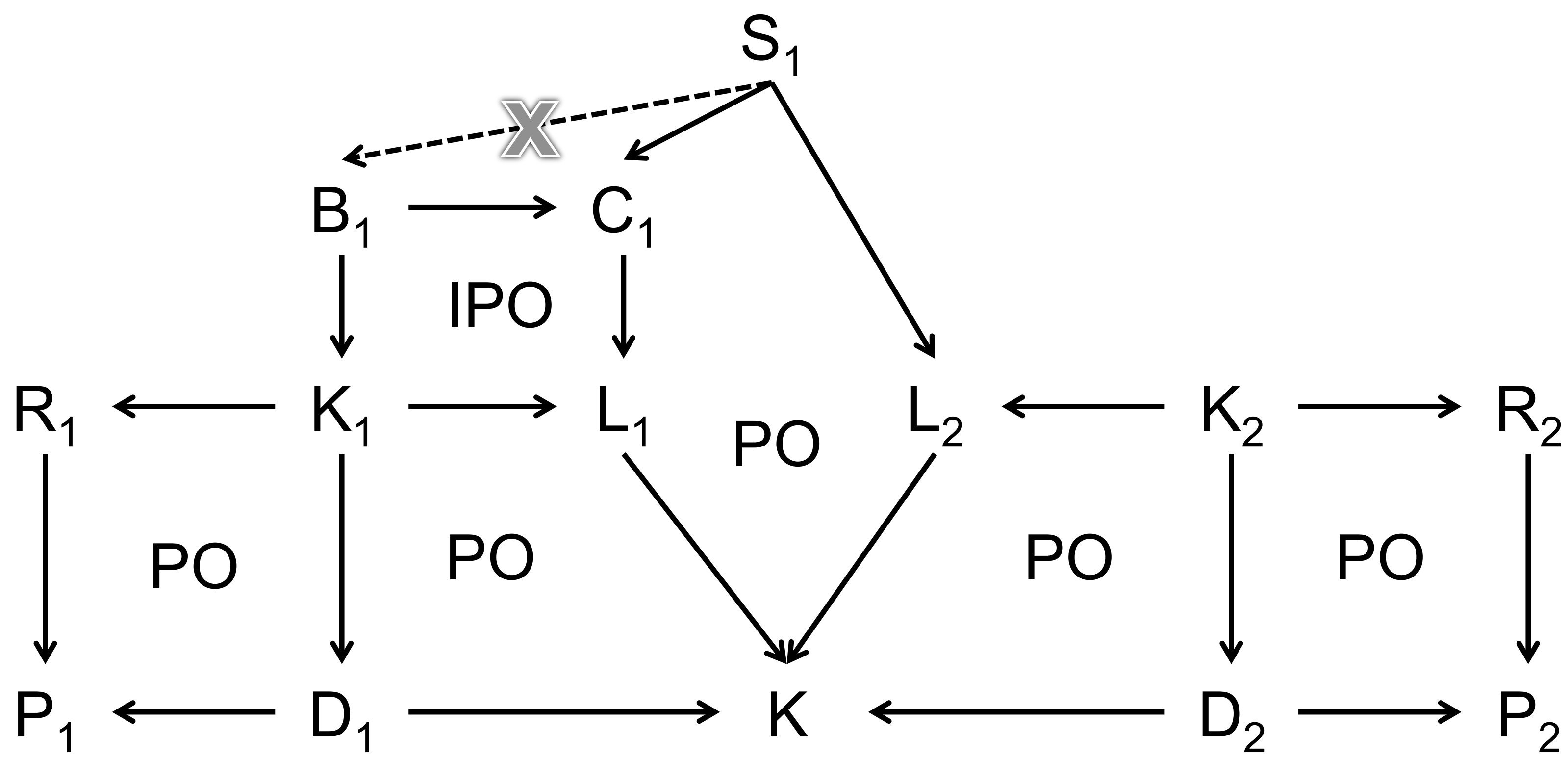
Conflict Reasons

Two transformations have a symmetrical delete-use conflict if

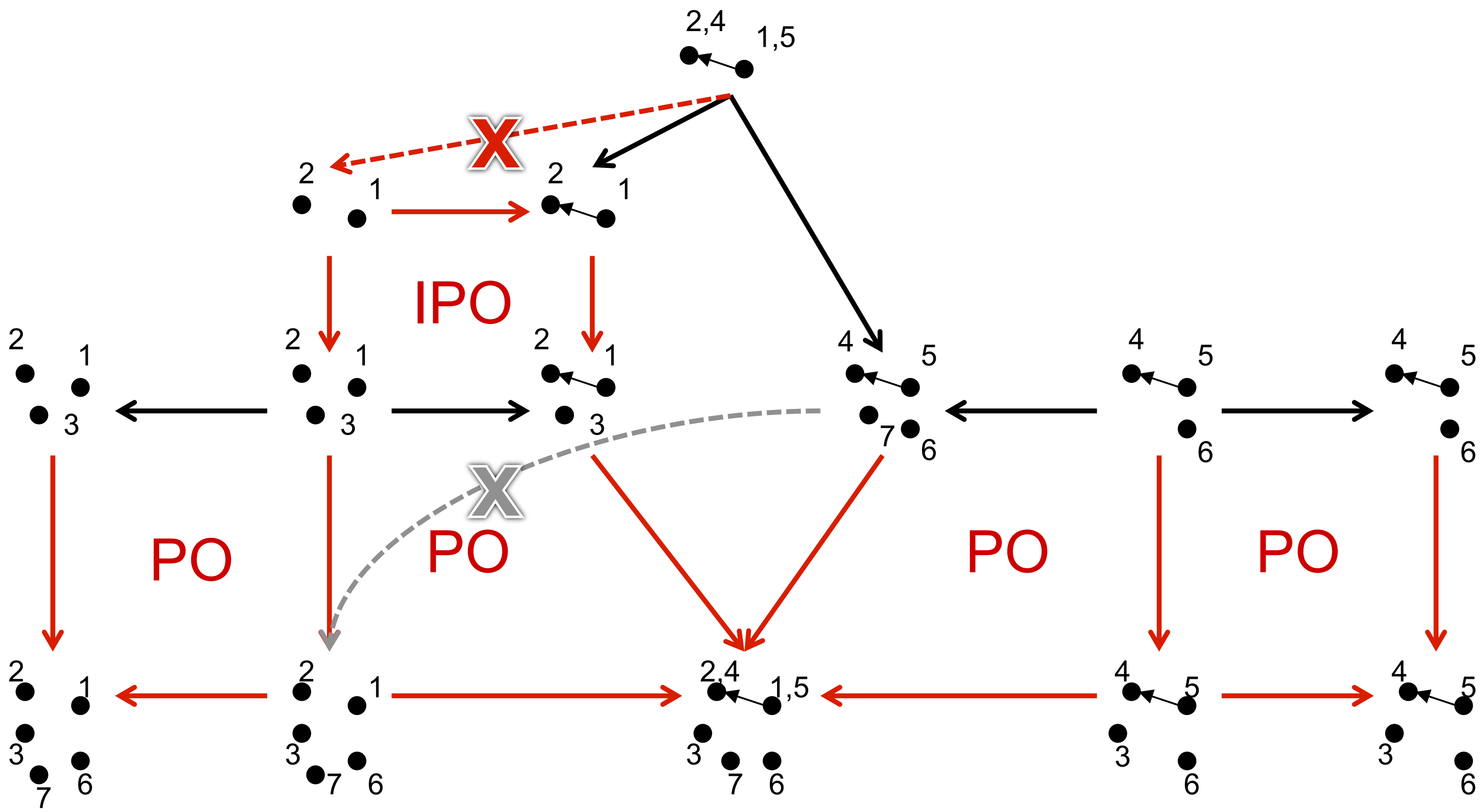


Essential Critical Pairs

The critical pair associated to a delete-use conflict is $P_1 \leftarrow K \rightarrow P_2$, where:

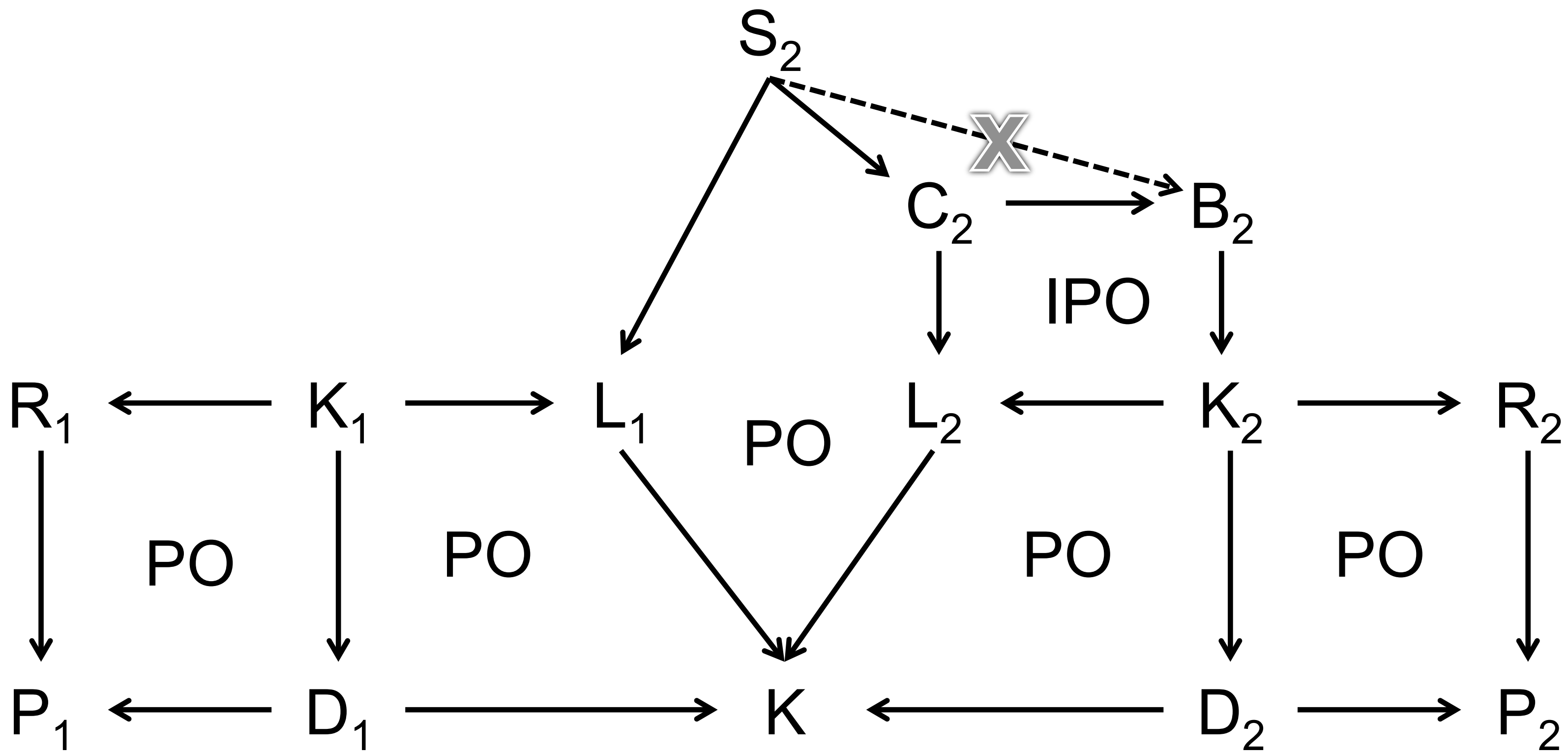


Example: Essential Critical Pair



Essential Critical Pairs

The critical pair associated to a use-delete conflict is $P_1 \leftarrow K \rightarrow P_2$, where:



Essential Critical Pairs

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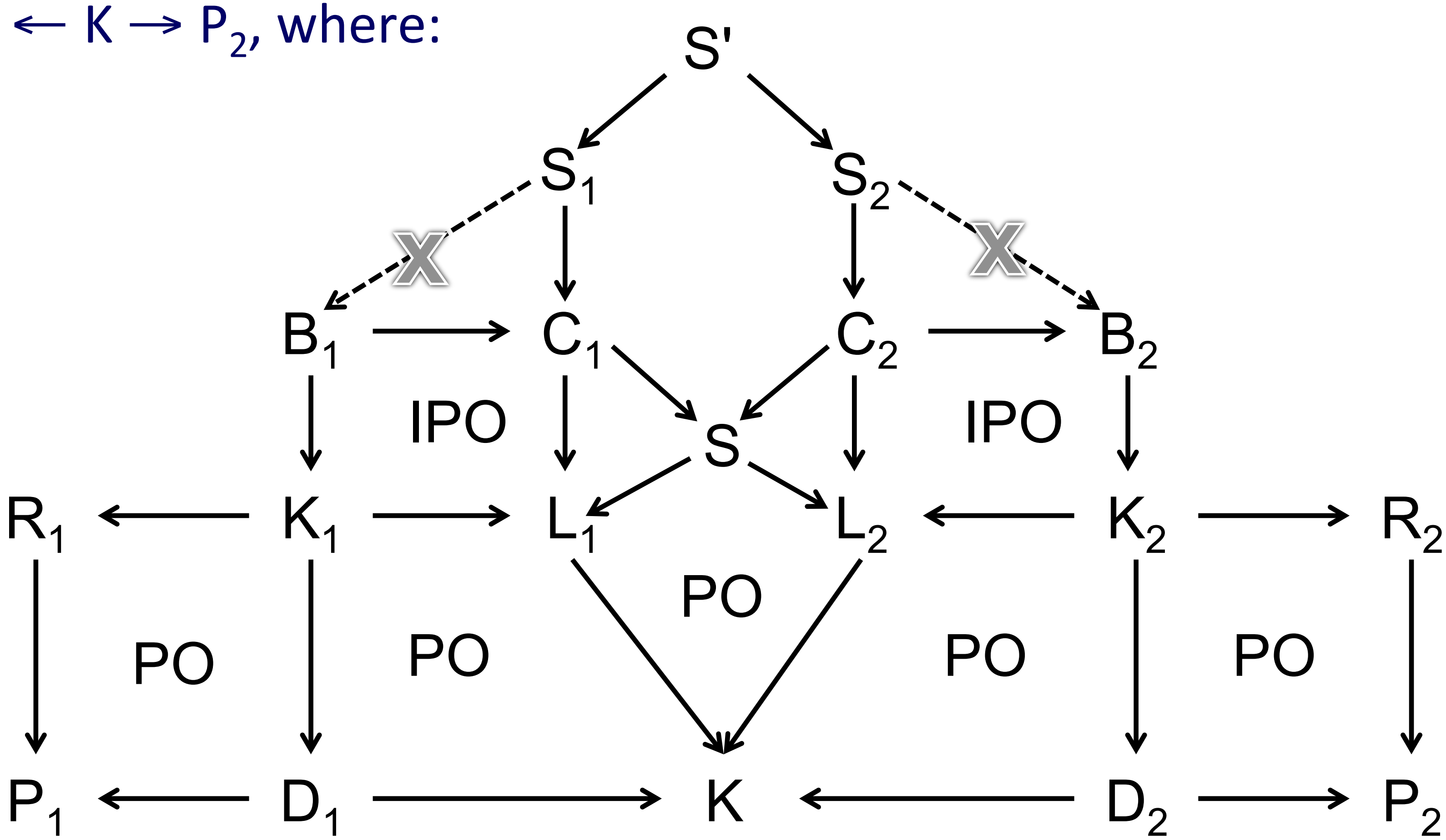


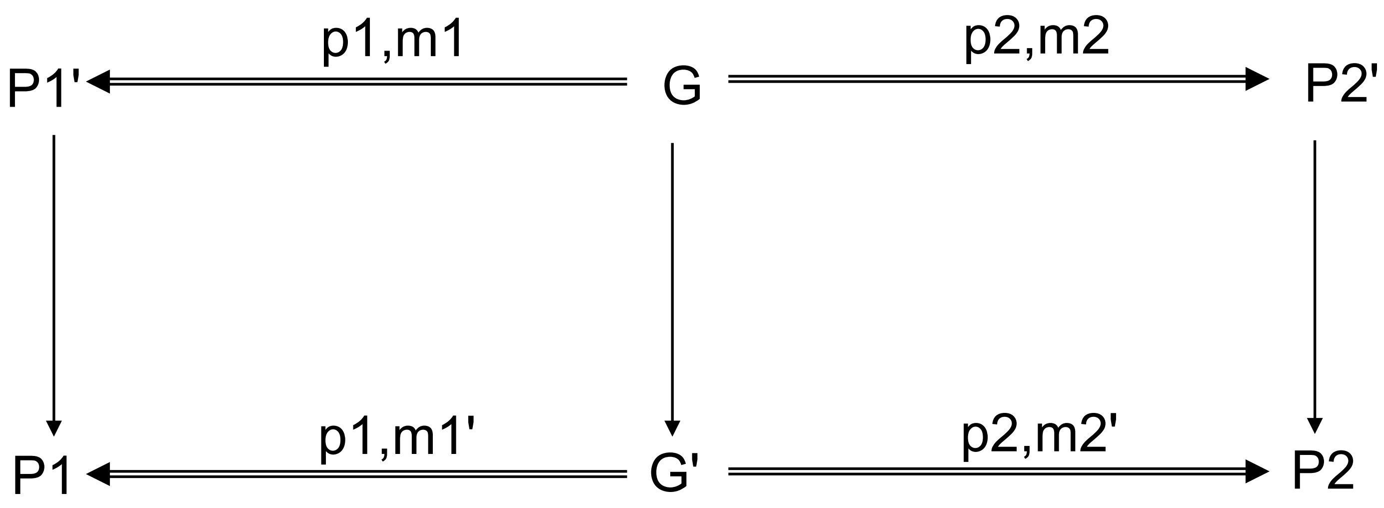
L. Lambers, H. Ehrig, and F. Orejas (2008)

Completeness Theorem

For any critical pair:

$$P1 \;_{p1,m1'}\!\Leftarrow G' \Rightarrow_{p2,m2'} P2$$

there is an essential critical pair $P1' \;_{p1,m1}\!\Leftarrow G \Rightarrow_{p2,m2} P2'$ with the same conflict reason, such that:

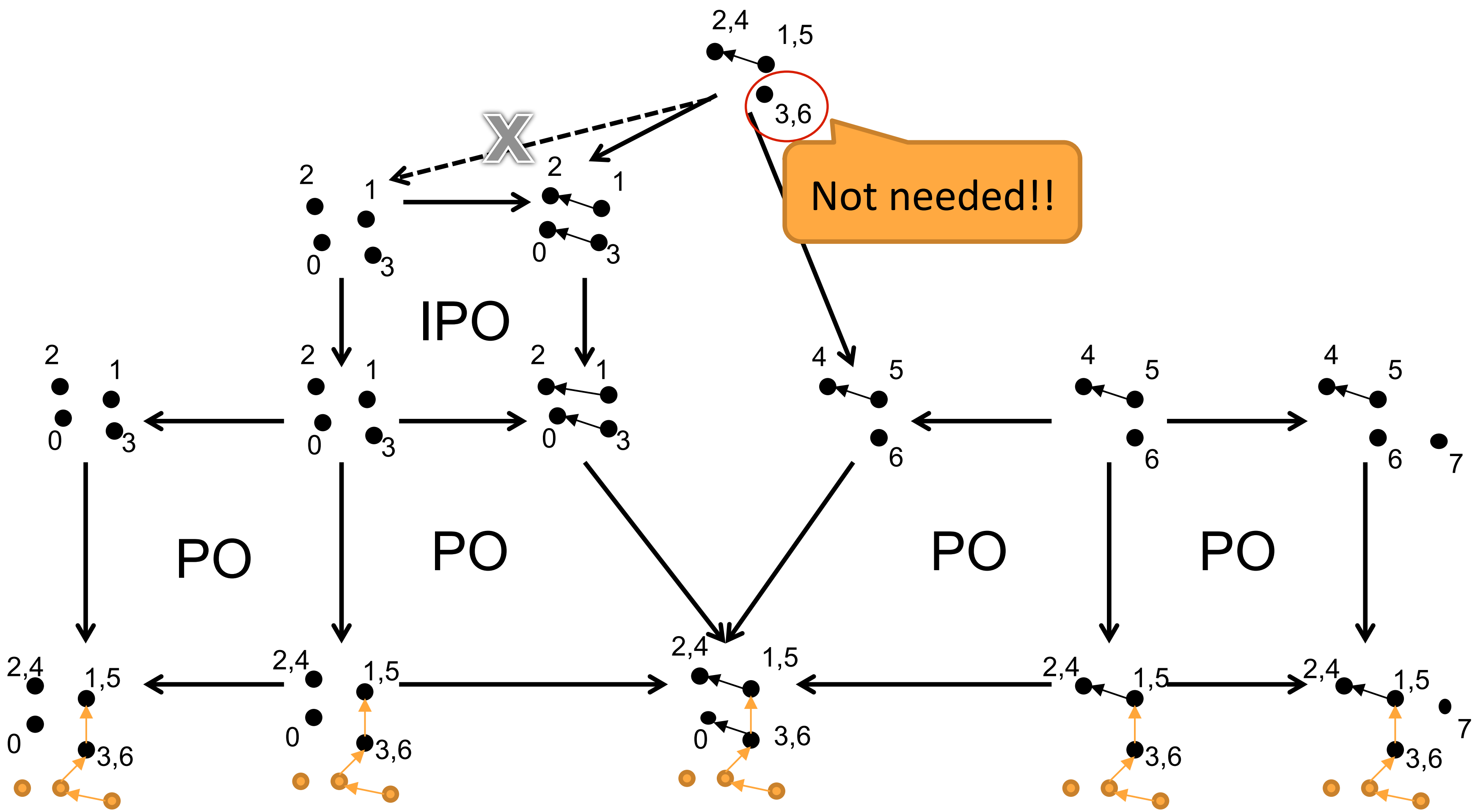


Local Confluence Theorem

A graph transformation system is locally confluent if all its essential critical pairs are strictly confluent.

Initial Critical Pairs

Conflict Reasons



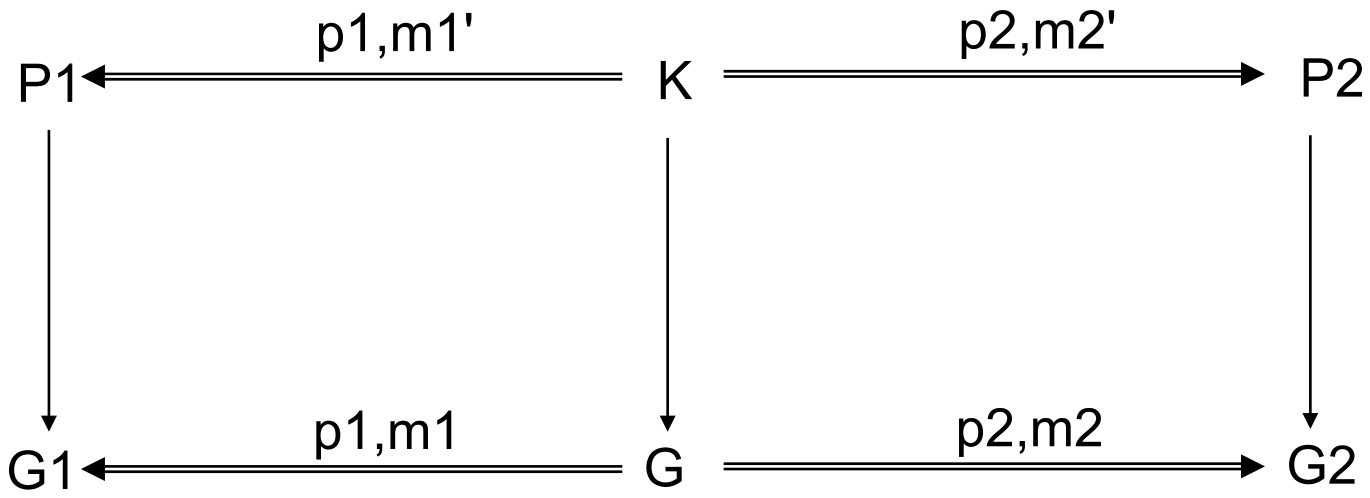
Initial Critical Pairs

Given direct transformations:

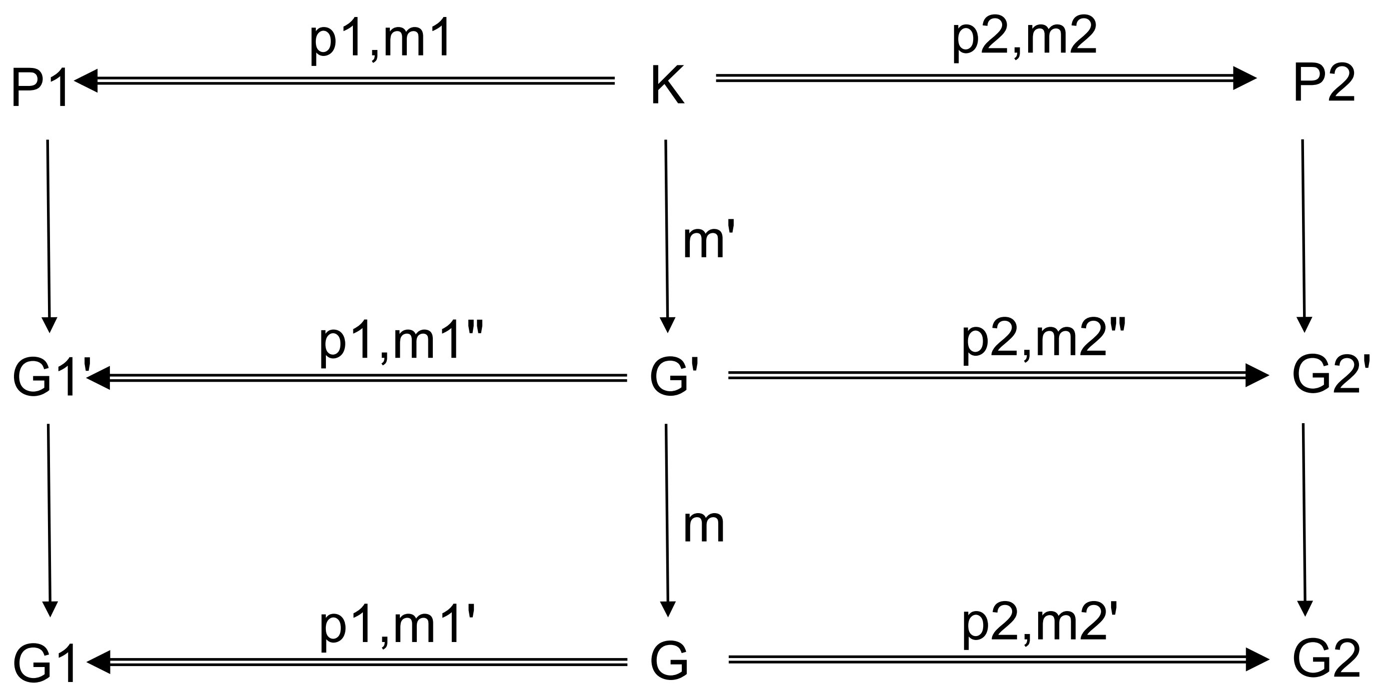
$$G1 \;{}_{p1,m1}\!\Leftarrow G \Rightarrow_{p2,m2} G2$$

$P1 \;{}_{p1,m1'}\!\Leftarrow K \Rightarrow_{p2,m2'} P2$ is an initial critical pair if:

1. There is an extension diagram



2. For any other pair of transformations G1' $_{p1,m1"}\Leftarrow$ G' $\Rightarrow_{p2,m2"}$ G2 that can be extended to G1 $_{p1,m1}\Leftarrow$ G $\Rightarrow_{p2,m2}$ G2, there is a unique morphism m': K → G', such that:



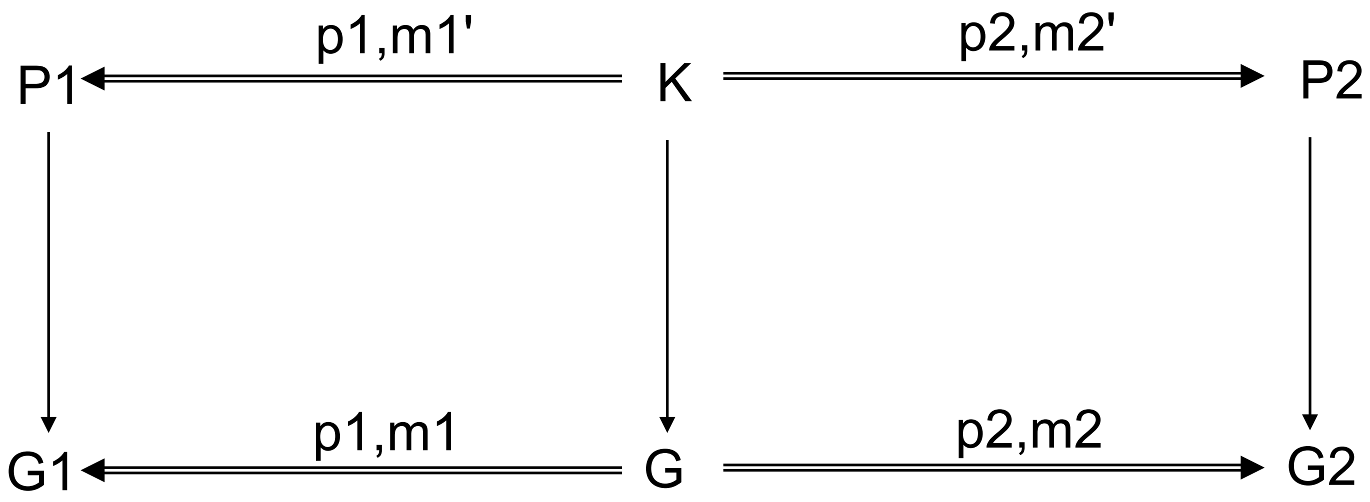
L. Lambers, K. Born, F. Orejas. D. Struber and G. Taentzer (2017)

Properties of Initial Critical Pairs

1. Initial critical pairs are essential critical pairs,
2. (Uniqueness) For any pair of transformations

$$G1 \; {}_{p1,m1}\!\Leftarrow G \Rightarrow_{p2,m2} G2$$

if it exists, there is a unique initial critical pair $P1_{p1,m1'}\!\Leftarrow K \Rightarrow_{p2,m2'} P2$ such that:



Existence of Initial Critical Pairs

- Initial critical pairs exist for typed graphs.

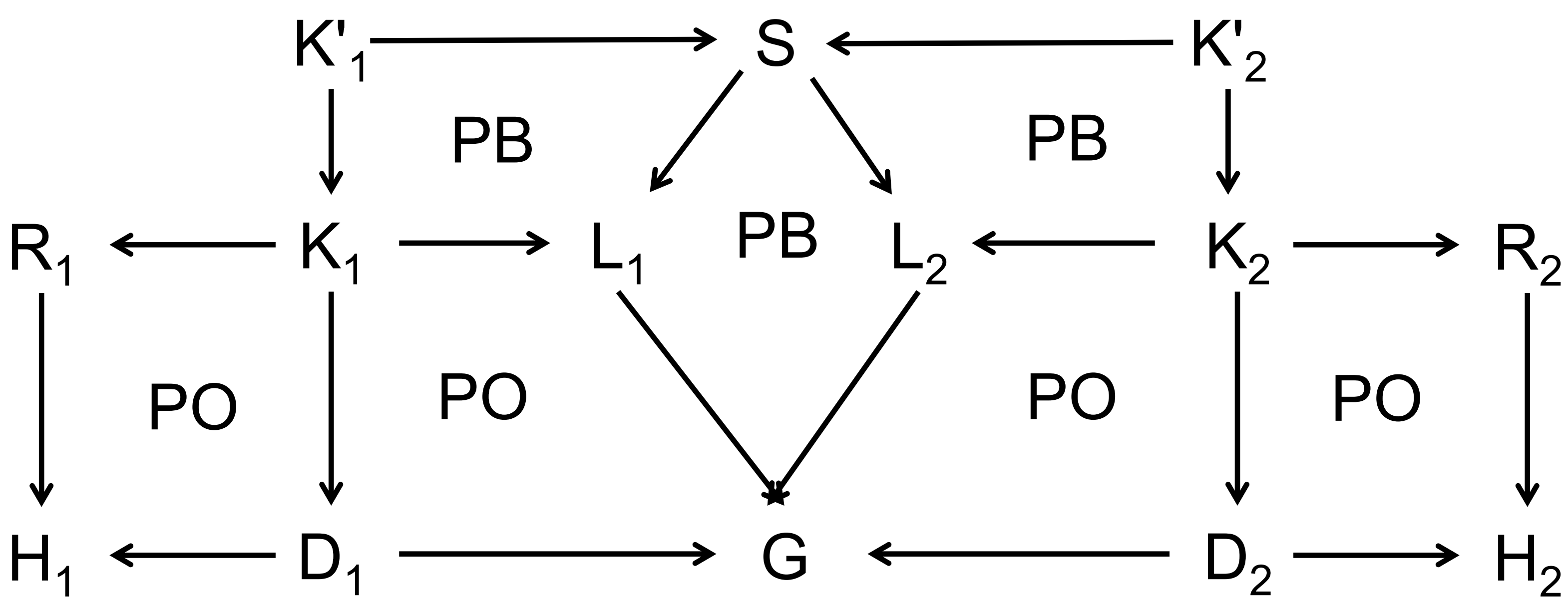
L. Lambers, K. Born, F. Orejas. D. Struber and G. Taentzer (2017)

- Initial critical pairs exist for arbitrary set-valued functor categories

G. G. Azzi, A. Corradini and L. Ribeiro (2018)

Parallel Independence

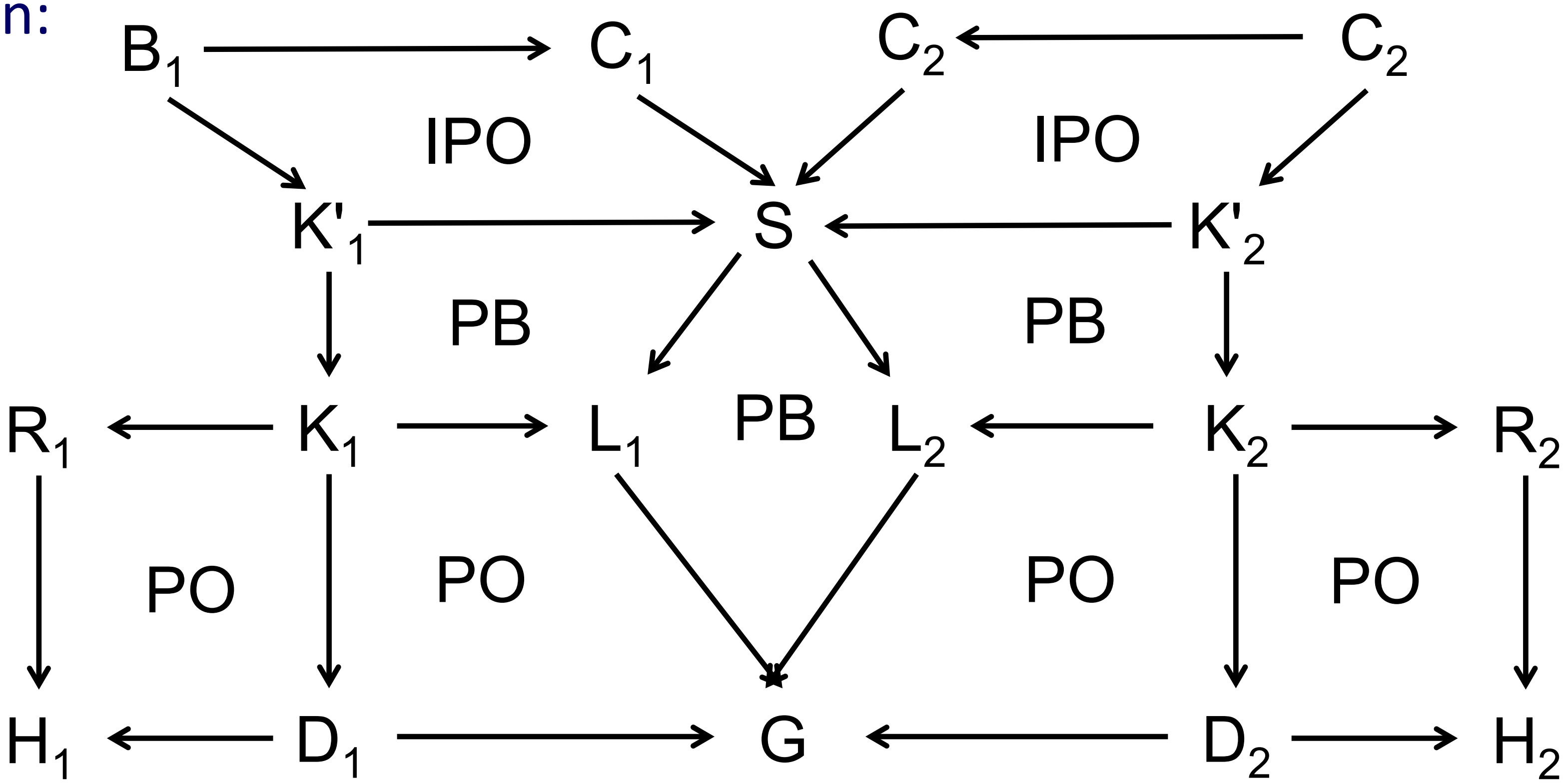
Two direct transformations are parallel independent if and only if the morphisms $K'_1 \to S$ and $K'_2 \to S$ are isomorphisms.



A. Corradini, D. Duval, M. Löwe, L. Ribeiro, R. Machado, A. Costa, G. G. Azzi, J.S. Bezerra, and L. M. Rodrigues (2018)

Construction of Initial Critical Pairs (Conflict Essences)

Given:

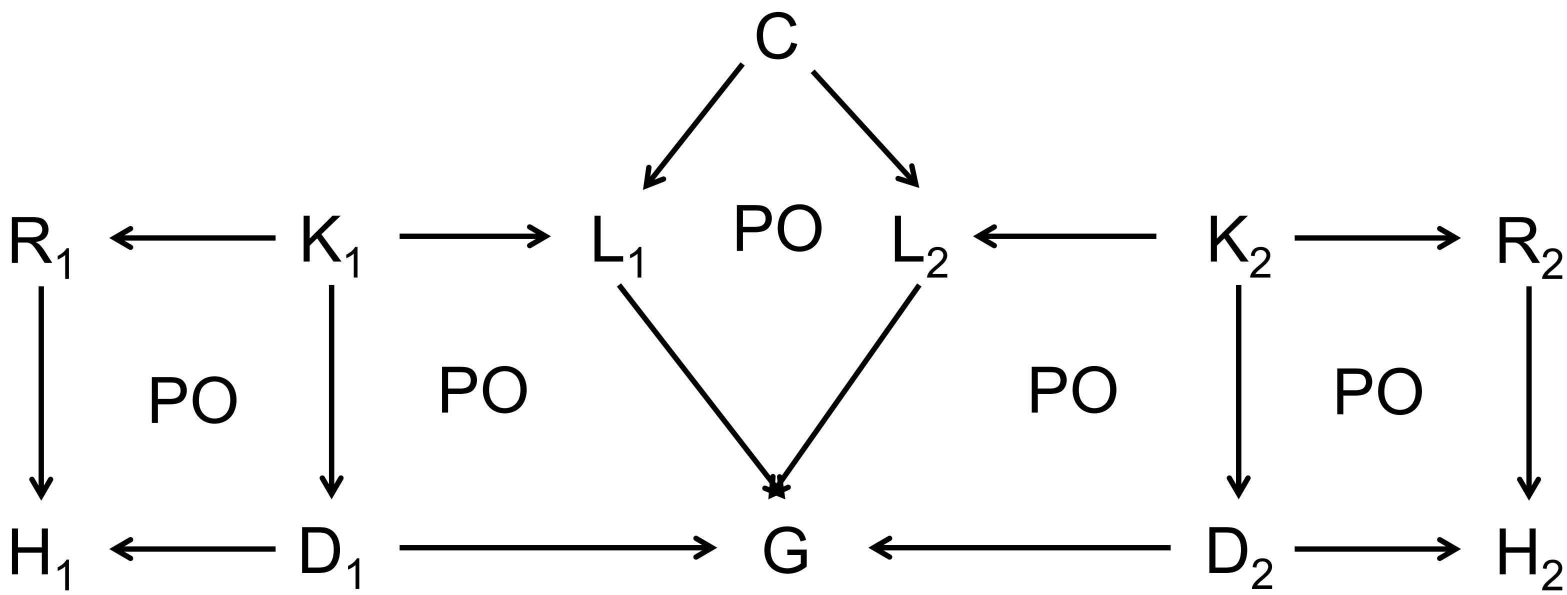


$C_1 \rightarrow S$ (resp. $C_2 \rightarrow S$) is a conflict essence if $K'_1 \rightarrow S$ (resp. $K'_2 \rightarrow S$) is not an isomorphism.

G. G. Azzi, A. Corradini and L. Ribeiro (2018)

Construction of Initial Critical Pairs

Given a conflict essence $C \rightarrow S$, its associated initial critical pair $P_1 \Leftarrow K \Rightarrow P_2$ is built as follows:



G. G. Azzi, A. Corradini and L. Ribeiro (2018)

Performance

No exhaustive performance comparison has been made, but a relatively simple example (a graph transformation implementation of a non-deterministic finite automaton) computed:

- 21,478 critical pairs
- 49 essential critical pairs
- 7 initial critical pairs

L. Lambers, K. Born, F. Orejas. D. Struber and G. Taentzer (2017)

Conclusion

We have seen several ways to check local confluence in graph transformation systems.

Some open problems:

- No efficient way to compute critical pairs for rules with NACs
- No proof that initial critical pairs exist for adhesive categories

¡Thank You!