

# Term Charters

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The evaluation  $\llbracket t \rrbracket_{(\Sigma, X)}(M, \beta) \in U_\Sigma(M)$  of a term  $t \in \mathcal{T}_\Sigma(X)$  formed over a signature  $\Sigma$  and value variables  $X$  generally extends a valuation  $\beta : X \rightarrow U_\Sigma(M)$  of the variables into the underlying values of a structure  $M$  such that the following conditions hold for variables  $x$ , variable renamings  $\xi : X_1 \rightarrow X_2$ , and signature morphisms  $\sigma : \Sigma \rightarrow \Sigma'$  (where  $-|\sigma$  denotes the reducts on value variables and structures, and  $\xi(t)$  resp.  $\sigma(t)$  the extension of  $\xi$  resp.  $\sigma$  to terms):

1. *Variables:*  $\llbracket x \rrbracket_{(\Sigma, X)}(M, \beta) = \beta(x)$ ;
2. *Substitutions:*  $\llbracket \xi(t) \rrbracket_{(\Sigma, X_2)}(M, \beta) = \llbracket t \rrbracket_{(\Sigma, X_1)}(M, \beta \circ \xi)$ ;
3. *Evaluations:*  $(\llbracket \sigma(t) \rrbracket_{(\Sigma', X')} (M', \beta') | \sigma = \llbracket t \rrbracket_{(\Sigma, X'|\sigma)} (M'|\sigma, \beta'|\sigma)$ .

Using indexed categories [3], we introduce *term charters* to give a general account of term evaluation over signatures, value variables, and structures based on an abstract formulation of conditions (1–3). In [1], we have already used a more complex version of term charters to demonstrate how sub-expression languages of the “Object Constraint Language” can be related and combined and how these languages give rise to *institutions*. Here<sup>3</sup>, we give a rather simplified account of term charters and we show that they provide a direct presentation of Pawlowski’s *context institutions* [2] which have been introduced to capture the notion of open formulæ over variables in institutions.

The general framework is built over a *term charter domain*  $(\mathbb{S}, Val, Str, U)$  consisting of a category  $\mathbb{S}$  of *signatures*, indexed categories  $Val, Str : \mathbb{S}^{\text{op}} \rightarrow \text{Cat}$  of *value variables* and *structures*, and an *underlying* indexed functor  $U : Str \rightarrow Val$ . For such a domain, let  $\mathcal{C} : Val \rightarrow Val$  be a lax indexed functor *constructing terms, renaming terms along value variable renamings, and translating terms along signature morphisms*, and  $\nu : 1_{Val} \rightarrow \mathcal{C}$  a lax indexed natural transformation *embedding value variables into terms*. Furthermore, for each  $\Sigma \in |\mathbb{S}|$ ,  $X \in |Val(\Sigma)|$ , and  $M \in |Str(\Sigma)|$ , let

$$(ext_\Sigma)_X^M : Val(\Sigma)(X, U_\Sigma(M)) \rightarrow Val(\Sigma)(\mathcal{C}_\Sigma(X), U_\Sigma(M))$$

be a function *extending a value variable valuation  $\beta$  into a term valuation  $(ext_\Sigma)_X^M(\beta)$* . Then  $(\mathcal{C}, \nu, ext)$  is a *term charter* over  $(\mathbb{S}, Val, Str, U)$  if the following requirements (V), (S), and (E) — directly corresponding to (1–3) — hold:

- (V)  $\nu_\Sigma(X); (ext_\Sigma)_X^M(\beta) = \beta$ ;
- (S)  $\mathcal{C}_\Sigma(\xi); (ext_\Sigma)_{X_2}^M(\beta) = (ext_\Sigma)_{X_1}^M(\xi; \beta)$ ;

<sup>3</sup> A paper draft is available at <https://www.informatik.uni-augsburg.de/en/chairs/swt/sse/publications/2018-Term-Charters.html>.

$$(E) \quad \mathcal{C}_\sigma(X'); \text{Val}(\sigma)((\text{ext}_{\Sigma'}^M)_{X'}^{M'}(\beta')) = (\text{ext}_\Sigma)^{\text{Str}(\sigma)(M')}_{\text{Val}(\sigma)(X')}(\text{Val}(\sigma)(\beta')).$$

On the one hand, this notion of term charters can be more compactly and succinctly characterised using  $\mathbb{S}$ -indexed comma categories where it is only required that  $\text{ext} : 1_{\text{Val}} \downarrow U \rightarrow \mathcal{C} \downarrow U$  is an indexed functor such that  $\text{ext}; (\nu \downarrow U) = 1_{1_{\text{Val}} \downarrow U}$  (with the indexed functor  $\nu \downarrow U : \mathcal{C} \downarrow U \rightarrow 1_{\text{Val}} \downarrow U$  given by  $(\nu \downarrow U)_\Sigma(X, \beta^\natural, M) = (X, \nu_\Sigma(X); \beta^\natural, M)$ ). On the other hand, applying the Grothendieck construction, we obtain the  $\mathcal{G}(\text{Val})$ -indexed category  $\text{Str}^\mathcal{G} = \mathcal{G}(1_{\text{Val}}) \downarrow U$  with a  $\mathcal{G}(\text{Val})$ -indexed functor  $\text{ext}^\mathcal{G} : \text{Str}^\mathcal{G} \rightarrow \mathcal{G}(\mathcal{C})^{\text{op}}; \text{Str}^\mathcal{G}$  given by  $\text{ext}_{\langle \Sigma, X \rangle}^\mathcal{G}(M, \beta) = (M, (\text{ext}_\Sigma)_X^M(\beta))$  such that  $\text{ext}^\mathcal{G}; (\mathcal{G}(\nu)^{\text{op}}; \text{Str}^\mathcal{G}) = 1_{\text{Str}^\mathcal{G}}$ ; we write  $|\text{ext}_{\langle \Sigma, X \rangle}^\mathcal{G}(M, \beta)|$  for  $(\text{ext}_\Sigma)_X^M(\beta)$ .

The Grothendieck presentation of a term charter  $\mathfrak{T} = (\mathcal{C}, \nu, \text{ext})$  yields an institution  $\mathfrak{I}^\mathfrak{T} = (\mathbb{S}^\mathfrak{T}, \text{Str}^\mathfrak{T}, \text{Sen}^\mathfrak{T}, \models^\mathfrak{T})$  if for each  $\Sigma \in |\mathbb{S}|$  there is a functor  $\mathcal{U}_\Sigma^* : \text{Val}(\Sigma) \rightarrow \text{Set}$  yielding *truth value variables* with a *truth value*  $*$   $\in \mathcal{U}_\Sigma^*(U_\Sigma(M))$  for each  $M \in |\text{Str}(\Sigma)|$  and  $\text{Val}(\sigma); \mathcal{U}_\Sigma^* = \mathcal{U}_{\Sigma'}^*$ , for all  $\sigma \in \mathbb{S}(\Sigma, \Sigma')$ : The category of signatures  $\mathbb{S}^\mathfrak{T}$  is defined to be  $\mathcal{G}(\text{Val})$ ; the indexed category  $\text{Str}^\mathfrak{T} : (\mathbb{S}^\mathfrak{T})^{\text{op}} \rightarrow \text{Cat}$  of structures as  $\text{Str}^\mathcal{G}$ ; the sentence functor  $\text{Sen}^\mathfrak{T} : \mathbb{S}^\mathfrak{T} \rightarrow \text{Set}$  as  $\text{Sen}^\mathfrak{T}(\langle \Sigma, X \rangle) = \mathcal{U}_\Sigma^*(\mathcal{C}_\Sigma(X))$  and  $\text{Sen}^\mathfrak{T}(\langle \sigma, \xi \rangle) = \mathcal{U}_{\Sigma'}^*(\mathcal{C}_\sigma(\xi))$ ; and the family of satisfaction relations  $(\models_{\langle \Sigma, X \rangle}^\mathfrak{T} \subseteq |\text{Str}^\mathfrak{T}(\langle \Sigma, X \rangle)| \times |\text{Sen}^\mathfrak{T}(\langle \Sigma, X \rangle)|)_{\langle \Sigma, X \rangle \in |\mathbb{S}^\mathfrak{T}|}$  by

$$(M, \beta) \models_{\langle \Sigma, X \rangle}^\mathfrak{T} \varphi \iff \mathcal{U}_\Sigma^*(|\text{ext}_{\langle \Sigma, X \rangle}^\mathcal{G}(M, \beta)|)(\varphi) = *.$$

Context institutions capture open formulæ over variables by *contexts*  $\text{Ctx}_\Sigma$  which directly correspond to  $\text{Val}(\Sigma)$  for a term charter domain. Context translations  $\text{Ctx}_\sigma : \text{Ctx}_\Sigma \rightarrow \text{Ctx}_{\Sigma'}$ , however, are handled covariantly rather than contravariantly as in term charters. If there is an adjunction  $(\eta_\sigma, \kappa_\sigma) : \sigma^{\text{Val}} \dashv \text{Val}(\sigma)$  to the value variable reduct, the naturality of  $\eta_\sigma$  yields the *coherence condition* of context institutions. Their *substitution* and *satisfaction conditions*

$$\begin{aligned} (M, \beta) \models_{\Sigma, X_2} \text{Frm}_\Sigma(\xi)(\varphi) &\iff (M, \xi; \beta) \models_{\Sigma, X_1} \varphi \\ (M', \beta') \models_{\Sigma', \text{Ctx}_\sigma(X)} \text{Frm}_{\sigma, X}(\varphi) &\iff (\text{Str}(\sigma)(M'), \sigma_{X, M'}^{\text{Val}}(\beta')) \models_{\Sigma, X} \varphi \end{aligned}$$

for the *formula functor*  $\text{Frm}_\Sigma : \text{Ctx}_\Sigma \rightarrow \text{Set}$  and the *formula translation*  $\text{Frm}_\sigma : \text{Frm}_\Sigma \rightarrow \text{Ctx}_\sigma; \text{Frm}_{\Sigma'}$  follow from (S) and (E), respectively.

## References

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