Term Charters

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The evaluation $[\![t]\!]_{(\Sigma,X)}(M,\beta) \in U_\Sigma(M)$ of a term $t \in \mathscr{T}_\Sigma(X)$ formed over a signature Σ and value variables X generally extends a valuation $\beta : X \to U_\Sigma(M)$ of the variables into the underlying values of a structure M such that the following conditions hold for variables x, variable renamings $\xi : X_1 \to X_2$, and signature morphisms $\sigma : \Sigma \to \Sigma'$ (where $-|\sigma$ denotes the reducts on value variables and structures, and $\xi(t)$ resp. $\sigma(t)$ the extension of ξ resp. σ to terms):

1. *Variables*: $[\![x]\!]_{(\Sigma,X)}(M,\beta) = \beta(x)$;
2. *Substitutions*: $[\![\xi(t)]\!]_{(\Sigma,X_2)}(M,\beta) = [\![t]\!]_{(\Sigma,X_1)}(M,\beta \circ \xi)$;
3. *Evaluations*: $([\![\sigma(t)]\!]_{(\Sigma',X')}(M',\beta'))|\sigma = [\![t]\!]_{(\Sigma,X'|\sigma)}(M'|\sigma,\beta'|\sigma)$.

Using indexed categories [3], we introduce *term charters* to give a general account of term evaluation over signatures, value variables, and structures based on an abstract formulation of conditions (1–3). In [1], we have already used a more complex version of term charters to demonstrate how sub-expression languages of the "Object Constraint Language" can be related and combined and how these languages give rise to *institutions*. Here[3], we give a rather simplified account of term charters and we show that they provide a direct presentation of Pawlowski's *context institutions* [2] which have been introduced to capture the notion of open formulæ over variables in institutions.

The general framework is built over a *term charter domain* $(\mathbb{S}, \mathit{Val}, \mathit{Str}, U)$ consisting of a category $\mathbb{S}$ of *signatures*, indexed categories $\mathit{Val}, \mathit{Str} : \mathbb{S}^{\mathrm{op}} \to \mathrm{Cat}$ of *value variables* and *structures*, and an *underlying* indexed functor $U : \mathit{Str} \dot{\to} \mathit{Val}$. For such a domain, let $\mathscr{C} : \mathit{Val} \ddot{\to} \mathit{Val}$ be a lax indexed functor *constructing terms, renaming terms along value variable renamings, and translating terms along signature morphisms*, and $\nu : 1_{\mathit{Val}} \dot{\to} \mathscr{C}$ a lax indexed natural transformation *embedding value variables into terms*. Furthermore, for each $\Sigma \in |\mathbb{S}|$, $X \in |\mathit{Val}(\Sigma)|$, and $M \in |\mathit{Str}(\Sigma)|$, let

$$(\mathit{ext}_\Sigma)^M_X : \mathit{Val}(\Sigma)(X, U_\Sigma(M)) \to \mathit{Val}(\Sigma)(\mathscr{C}_\Sigma(X), U_\Sigma(M))$$

be a function *extending a value variable valuation β into a term valuation* $(\mathit{ext}_\Sigma)^M_X(\beta)$. Then $(\mathscr{C}, \nu, \mathit{ext})$ is a *term charter* over $(\mathbb{S}, \mathit{Val}, \mathit{Str}, U)$ if the following requirements (V), (S), and (E) — directly corresponding to (1–3) — hold:

(V) $\nu_\Sigma(X); (\mathit{ext}_\Sigma)^M_X(\beta) = \beta$;

(S) $\mathscr{C}_\Sigma(\xi); (\mathit{ext}_\Sigma)^M_{X_2}(\beta) = (\mathit{ext}_\Sigma)^M_{X_1}(\xi; \beta)$;

[3] A paper draft is available at https://www.informatik.uni-augsburg.de/en/chairs/swt/sse/publications/2018-Term-Charters.html.

(E) $\mathscr{C}_\sigma(X'); \mathit{Val}(\sigma)((\mathit{ext}_{\Sigma'})^{M'}_{X'}(\beta')) = (\mathit{ext}_\Sigma)^{\mathit{Str}(\sigma)(M')}_{\mathit{Val}(\sigma)(X')}(\mathit{Val}(\sigma)(\beta'))$.

On the one hand, this notion of term charters can be more compactly and succinctly characterised using $\mathbb{S}$-indexed comma categories where it is only required that $\mathit{ext} : 1_{\mathit{Val}} \downarrow U \dot{\to} \mathscr{C} \downarrow U$ is an indexed functor such that $\mathit{ext}; (\nu \downarrow U) = 1_{1_{\mathit{Val}} \downarrow U}$ (with the indexed functor $\nu \downarrow U : \mathscr{C} \downarrow U \dot{\to} 1_{\mathit{Val}} \downarrow U$ given by $(\nu \downarrow U)_\Sigma(X, \beta^\natural, M) = (X, \nu_\Sigma(X); \beta^\natural, M)$). On the other hand, applying the Grothendieck construction, we obtain the $\mathcal{G}(\mathit{Val})$-indexed category $\mathit{Str}^{\mathcal{G}} = \mathcal{G}(1_{\mathit{Val}}) \downarrow U$ with a $\mathcal{G}(\mathit{Val})$-indexed functor $\mathit{ext}^{\mathcal{G}} : \mathit{Str}^{\mathcal{G}} \dot{\to} \mathcal{G}(\mathscr{C})^{\mathrm{op}}; \mathit{Str}^{\mathcal{G}}$ given by $\mathit{ext}^{\mathcal{G}}_{\langle \Sigma, X\rangle}(M, \beta) = (M, (\mathit{ext}_\Sigma)^M_X(\beta))$ such that $\mathit{ext}^{\mathcal{G}}; (\mathcal{G}(\nu)^{\mathrm{op}}; \mathit{Str}^{\mathcal{G}}) = 1_{\mathit{Str}^{\mathcal{G}}}$; we write $|\mathit{ext}^{\mathcal{G}}_{\langle \Sigma, X\rangle}(M, \beta)|$ for $(\mathit{ext}_\Sigma)^M_X(\beta)$.

The Grothendieck presentation of a term charter $\mathfrak{T} = (\mathscr{C}, \nu, \mathit{ext})$ yields an institution $\mathfrak{I}^{\mathfrak{T}} = (\mathbb{S}^{\mathfrak{T}}, \mathit{Str}^{\mathfrak{T}}, \mathit{Sen}^{\mathfrak{T}}, \models^{\mathfrak{T}})$ if for each $\Sigma \in |\mathbb{S}|$ there is a functor $\mathcal{U}^*_\Sigma : \mathit{Val}(\Sigma) \to \mathrm{Set}$ yielding *truth value variables* with a *truth value* $* \in \mathcal{U}^*_\Sigma(U_\Sigma(M))$ for each $M \in |\mathit{Str}(\Sigma)|$ and $\mathit{Val}(\sigma); \mathcal{U}^*_\Sigma = \mathcal{U}^*_{\Sigma'}$ for all $\sigma \in \mathbb{S}(\Sigma, \Sigma')$: The category of signatures $\mathbb{S}^{\mathfrak{T}}$ is defined to be $\mathcal{G}(\mathit{Val})$; the indexed category $\mathit{Str}^{\mathfrak{T}} : (\mathbb{S}^{\mathfrak{T}})^{\mathrm{op}} \to \mathrm{Cat}$ of structures as $\mathit{Str}^{\mathcal{G}}$; the sentence functor $\mathit{Sen}^{\mathfrak{T}} : \mathbb{S}^{\mathfrak{T}} \to \mathrm{Set}$ as $\mathit{Sen}^{\mathfrak{T}}(\langle \Sigma, X\rangle) = \mathcal{U}^*_\Sigma(\mathscr{C}_\Sigma(X))$ and $\mathit{Sen}^{\mathfrak{T}}(\langle \sigma, \xi\rangle) = \mathcal{U}^*_\Sigma(\mathscr{C}_\sigma(\xi))$; and the family of satisfaction relations $(\models^{\mathfrak{T}}_{\langle \Sigma, X\rangle} \subseteq |\mathit{Str}^{\mathfrak{T}}(\langle \Sigma, X\rangle)| \times |\mathit{Sen}^{\mathfrak{T}}(\langle \Sigma, X\rangle)|)_{\langle \Sigma, X\rangle \in |\mathbb{S}^{\mathfrak{T}}|}$ by

$$(M, \beta) \models^{\mathfrak{T}}_{\langle \Sigma, X\rangle} \varphi \iff \mathcal{U}^*_\Sigma(|\mathit{ext}^{\mathcal{G}}_{\langle \Sigma, X\rangle}(M, \beta)|)(\varphi) = * \,.$$

Context institutions capture open formulæ over variables by *contexts* Ctxt_Σ which directly correspond to $\mathit{Val}(\Sigma)$ for a term charter domain. Context translations $\mathit{Ctxt}_\sigma : \mathrm{Ctxt}_\Sigma \to \mathrm{Ctxt}_{\Sigma'}$, however, are handled covariantly rather than contravariantly as in term charters. If there is an adjunction $(\eta_\sigma, \kappa_\sigma) : \sigma^{\mathit{Val}} \dashv \mathit{Val}(\sigma)$ to the value variable reduct, the naturality of η_σ yields the *coherence condition* of context institutions. Their *substitution* and *satisfaction conditions*

$$(M, \beta) \models_{\Sigma, X_2} \mathit{Frm}_\Sigma(\xi)(\varphi) \iff (M, \xi; \beta) \models_{\Sigma, X_1} \varphi$$
$$(M', \beta') \models_{\Sigma', \mathit{Ctxt}_\sigma(X)} \mathit{Frm}_{\sigma, X}(\varphi) \iff (\mathit{Str}(\sigma)(M'), \sigma^{\mathit{Val}}_{X, M'}(\beta')) \models_{\Sigma, X} \varphi$$

for the *formula functor* $\mathit{Frm}_\Sigma : \mathrm{Ctxt}_\Sigma \to \mathrm{Set}$ and the *formula translation* $\mathit{Frm}_\sigma : \mathit{Frm}_\Sigma \dot{\to} \mathit{Ctxt}_\sigma; \mathit{Frm}_{\Sigma'}$ follow from (S) and (E), respectively.

References

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